Curvature

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1 Symmetries of the Riemann tensor

The Riemann tensor is the curvature tensor of the Levi-Civita connection. It is defined as

$$\boldsymbol{R} : \begin{array}{c} \Gamma(\mathcal{M})^3 \to \Gamma(\mathcal{M}) \\ (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w} \end{array} \tag{1}$$

with $\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v}) = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}$. Its components over the coordinate basis $\{\boldsymbol{\partial}_\mu\}$ are

$$\boldsymbol{R}(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu)\boldsymbol{\partial}_\sigma = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\boldsymbol{\partial}_\sigma = R^\rho{}_{\sigma\mu\nu}\boldsymbol{\partial}_\rho. \tag{2}$$

Q1. Show that $R^\sigma{}_{\rho\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\rho\nu} - \partial_\nu\Gamma^\sigma{}_{\rho\mu} + \Gamma^\sigma{}_{\tau\mu}\Gamma^\tau{}_{\rho\nu} - \Gamma^\sigma{}_{\tau\nu}\Gamma^\tau{}_{\rho\mu}$.

The components of the Riemann tensor enjoy a number of symmetry, namely

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \qquad \text{(definition)}, \tag{3}$$

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \qquad (\nabla \text{ is metric preserving}), \tag{4}$$

$$R_{\mu[\nu\rho\sigma]} = 0 \qquad (\nabla \text{ is torsion free}). \tag{5}$$

Q2. Show that, given eqs. (3), (4),

$$(5) \iff \begin{cases} R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \\ R_{[\mu\nu\rho\sigma]} = 0 \end{cases} \tag{6}$$

Hint for ($\Leftarrow$): check that if $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$, then $R_{\mu[\nu\rho\sigma]} = -R_{\nu[\mu\rho\sigma]} = -R_{\rho[\nu\mu\sigma]} = -R_{\sigma[\nu\rho\mu]}$.

Q3. Deduce from those symmetries that the number of independent and non-zero components of the Riemann tensor in dimension d is

$$N = \frac{d^2(d^2-1)}{12}. \tag{7}$$

Hint: There are several ways to prove this; one of them leads to

$$N = \frac{\frac{d(d-1)}{2}\left[\frac{d(d-1)}{2} + 1\right]}{2} - \binom{d}{4}.$$

The Ricci tensor $\boldsymbol{Ric} : (\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{Ric}(\boldsymbol{u}, \boldsymbol{v}) = R_{\mu\nu}u^\mu v^\nu$ is a form of trace of the Riemann tensor, in the sense that its components read $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$.

Q4. How many independent components does the Ricci tensor have in dimension d? Conclude that the Ricci tensor contains all the information about Riemann curvature for $d \leq 3$.

2 An alternative definition for the covariant derivative

Suppose a Riemannian manifold, $\mathcal{M}$, is embedded into Euclidean space ($\mathbb{R}^n$) via the mapping $\vec{\Psi} : \mathbb{R}^d \supset U \to \mathbb{R}^n$ such that the tangent space at $\vec{\Psi}(P)$ is spanned by the vectors

$$\left\{ \left. \frac{\partial \vec{\Psi}}{\partial x^i} \right|_P : i \in \{1, ..., d\} \right\} \tag{8}$$

and the scalar product on $\mathbb{R}^n$ is compatible with the metric on $\mathcal{M}$:

$$g_{ij} = \left\langle \frac{\partial \vec{\Psi}}{\partial x^i}, \frac{\partial \vec{\Psi}}{\partial x^j} \right\rangle . \tag{9}$$

Note that 8 is simply the basis of the tangent vector space at point P. Here, d is the dimension of the manifold with $d < n$.

Q1. The (contravariant) derivative of a metric g_{ab} is given by

$$\frac{\partial g_{ab}}{\partial x^c} = \left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^c \, \partial x^a}, \frac{\partial \vec{\Psi}}{\partial x^b} \right\rangle + \left\langle \frac{\partial \vec{\Psi}}{\partial x^a}, \frac{\partial^2 \vec{\Psi}}{\partial x^c \, \partial x^b} \right\rangle \tag{10}$$

Using 10, show that

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} = 2 \left\langle \frac{\partial \vec{\Psi}}{\partial x^k}, \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j} \right\rangle . \tag{11}$$

A vector field ($\vec{V}$) in the tangent vector space of the manifold $\mathcal{M}$ can be written as

$$\vec{V} = v^j \frac{\partial \vec{\Psi}}{\partial x^j} \tag{12}$$

where, as mentioned before, $\partial \vec{\Psi} / \partial x^j$ is the basis of the tangent vector space. One has

$$\frac{\partial \vec{V}}{\partial x^i} = \frac{\partial v^j}{\partial x^i} \frac{\partial \vec{\Psi}}{\partial x^j} + v^j \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j} \tag{13}$$

Q2. Do both the terms in 13 belong to the tangent space?

The second term in 13 can be expressed as

$$\frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j} = \Gamma^k{}_{ij} \frac{\partial \vec{\Psi}}{\partial x^k} + \vec{n} \tag{14}$$

This is a linear combination of the tangent space base vectors (using the Christoffel symbols as linear factors) and a vector orthogonal to the tangent space. The covariant derivative $\nabla_{\mathbf{e}_i} \vec{V}$, also written as $\nabla_i \vec{V}$, is defined as the orthogonal projection of the usual derivative onto tangent space:

$$\nabla_{\mathbf{e}_i} \vec{V} := \frac{\partial \vec{V}}{\partial x^i} - \vec{n} = \left(\frac{\partial v^k}{\partial x^i} + v^j \Gamma^k{}_{ij} \right) \frac{\partial \vec{\Psi}}{\partial x^k} . \tag{15}$$

Since $\vec{n}$ is orthogonal to tangent space, one can solve the normal equations:

$$\left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j}, \frac{\partial \vec{\Psi}}{\partial x^l} \right\rangle = \Gamma^k{}_{ij} \left\langle \frac{\partial \vec{\Psi}}{\partial x^k}, \frac{\partial \vec{\Psi}}{\partial x^l} \right\rangle = \Gamma^k{}_{ij} \, g_{kl} \tag{16}$$

Q3. Using 11 and 16, write the expression for the Christoffel symbols.

Q4. Why do you think it is more convenient to define the covariant derivative the way you did in class?

3 Sphere and cylinder

Consider a sphere with radius r embedded in the three-dimensional Euclidean space. This exercise proposes to study the geometric properties of the surface of the sphere, as a curved two-dimensional manifold. We will describe the surface of the sphere using spherical coordinates θ, φ.

Q1. What is the distance between two points of the sphere whose coordinates are (θ, φ) and $(\theta + \mathrm{d}\theta, \varphi + \mathrm{d}\varphi)$? Deduce from this the expression of the metric of the sphere, as

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}\theta^a\mathrm{d}\theta^b = g_{\theta\theta}\mathrm{d}\theta^2 + 2g_{\theta\varphi}\mathrm{d}\theta\mathrm{d}\varphi + g_{\varphi\varphi}\mathrm{d}\varphi^2. \tag{17}$$

Q2. Determine the Christoffel symbols associated with this metric.

Q3. Identify the only non-zero component of the Riemann tensor, and calculate its value.

Q4. Calculate the Ricci scalar, and compare with the quantity that you would expect for the curvature of a sphere.

Consider a cylinder with radius r about the z-axis of the three-dimensional Euclidean space.

Q5. Show that the metric on the surface of the cylinder reads

$$\mathrm{d}s^2 = r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2. \tag{18}$$

Q6. Determine the Riemann tensor for this surface. What do you think about this result?