# Connection and Metric

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## 1 Connection and covariant derivative

A connection $\nabla$ on a manifold $\mathcal{M}$ is a structure which allows one to transport vectors, forms, and tensors from event to event. Let $\boldsymbol{u}$ be a vector field over $\mathcal{M}$, then $\nabla_{\boldsymbol{u}}$ is called the *covariant derivative* along $\boldsymbol{u}$, and can act on functions, vectors, forms, and tensors.

The effect of $\nabla_{\boldsymbol{u}}$ on a function $f$ is simply

$$\nabla_{\boldsymbol{u}} f \equiv \boldsymbol{u}(f) \tag{1}$$

that is, the derivative of $f$ along $\boldsymbol{u}$. The action of $\nabla_{\boldsymbol{u}}$ on vectors satisfies the following properties: for any function $f$ and any vectors fields $\boldsymbol{v}, \boldsymbol{w}$,

$$\nabla_{\boldsymbol{u}+f\boldsymbol{v}} \boldsymbol{w} = \nabla_{\boldsymbol{u}} \boldsymbol{w} + f \nabla_{\boldsymbol{v}} \boldsymbol{w}, \tag{2}$$

$$\nabla_{\boldsymbol{u}} (\boldsymbol{v} + \boldsymbol{w}) = \nabla_{\boldsymbol{u}} \boldsymbol{v} + \nabla_{\boldsymbol{u}} \boldsymbol{w}, \tag{3}$$

$$\nabla_{\boldsymbol{u}} (f\boldsymbol{v}) = \boldsymbol{u}(f)\boldsymbol{v} + f \nabla_{\boldsymbol{u}} \boldsymbol{v}. \tag{4}$$

If we restrict to the action on vectors, $\nabla$ can thus be seen as a function which eats two vectors and returns a vector, $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \nabla_{\boldsymbol{u}} \boldsymbol{v}$.

**Q1.** From this point of view, is $\nabla$ a tensor? Why?

Let $\{x^\mu\}$ be a coordinate system on $\mathcal{M}$. One can define a notion of components for $\nabla$ with respect to the coordinate basis $\{\boldsymbol{\partial}_\mu\}$, as

$$\nabla_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu \equiv \Gamma^\rho{}_{\nu\mu} \boldsymbol{\partial}_\rho, \tag{5}$$

where the numbers $\Gamma^\rho{}_{\mu\nu}$ are called the *connection coefficients*. The covariant derivative with respect to the coordinate basis $\nabla_{\boldsymbol{\partial}_\mu}$ is usually denoted $\nabla_\mu$ for short.

**Q2.** Show that the covariant derivative of a vector field reads

$$\nabla_\mu \boldsymbol{v} = \left( \partial_\mu v^\nu + \Gamma^\nu{}_{\rho\mu} v^\rho \right) \boldsymbol{\partial}_\nu. \tag{6}$$

The component $(\nabla_\mu \boldsymbol{v})^\nu$ is usually denoted $\nabla_\mu v^\nu$ for short.

**Q3.** How do the coefficients $\Gamma^\rho{}_{\nu\mu}$ change under a coordinate transformation $\{x^\mu\} \to \{y^\alpha\}$?

The action of the covariant derivative $\nabla_{\boldsymbol{u}}$ can be extended to differential forms. For that purpose, we assume that if $\boldsymbol{\omega}$ is a one-form, then $\nabla_{\boldsymbol{u}} \boldsymbol{\omega}$ is also a one-form, with

$$\forall \boldsymbol{v} \in \Gamma(\mathcal{M}) \qquad \nabla_{\boldsymbol{u}} [\boldsymbol{\omega}(\boldsymbol{v})] = (\nabla_{\boldsymbol{u}} \boldsymbol{\omega})(\boldsymbol{v}) + \boldsymbol{\omega}(\nabla_{\boldsymbol{u}} \boldsymbol{v}). \tag{7}$$

**Q4.** Derive the expression for the components $\nabla_\mu \omega_\nu$, a short-hand notation for $(\nabla_\mu \boldsymbol{\omega})_\nu$. Deduce the expression of $\nabla_\mu \mathbf{d}x^\sigma$.

The action of $\nabla_{\boldsymbol{u}}$ can even be further extended to tensors, assuming that for any tensors $\boldsymbol{X}, \boldsymbol{Y}$,

$$\nabla_{\boldsymbol{u}}(\boldsymbol{X} \otimes \boldsymbol{Y}) = (\nabla_{\boldsymbol{u}}\boldsymbol{X}) \otimes \boldsymbol{Y} + \boldsymbol{X} \otimes (\nabla_{\boldsymbol{u}}\boldsymbol{Y}). \tag{8}$$

**Q5.** Show that for any rank-$(m, n)$ tensor:

$$\begin{aligned}
\nabla_\rho X^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} &\equiv (\nabla_\rho \boldsymbol{X})^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} & (9)\\
&= \partial_\rho X^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} + \Gamma^{\mu_1}{}_{\sigma\rho} X^{\sigma \dots \mu_n}{}_{\nu_1 \dots \nu_m} + \dots + \Gamma^{\mu_n}{}_{\sigma\rho} X^{\mu_1 \dots \sigma}{}_{\nu_1 \dots \nu_m}\\
&\qquad - \Gamma^{\sigma}{}_{\nu_1\rho} X^{\mu_1 \dots \mu_n}{}_{\sigma \dots \nu_m} - \dots - \Gamma^{\sigma}{}_{\nu_m\rho} X^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \sigma}. & (10)
\end{aligned}$$

## 2 Metric and inverse metric

In Riemannian (or pseudo-Riemannian) geometry, the manifold $\mathcal{M}$ is not only equipped with a connection, but also with a metric tensor $\boldsymbol{g} = g_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$, which allows one to define the *scalar product* between two vectors $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu}u^\mu v^\nu$, and hence angles and distances.

**Q1.** How does the metric coefficients change under a coordinate transformation $\{x^\mu\} \to \{y^\alpha\}$?

**Q2.** Consider the metric given by the line element

$$\begin{aligned}
\mathrm{d}s^2 &\equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu & (11)\\
&= -(1 - \Omega^2 r^2 \sin^2\theta)\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + 2\Omega r^2 \sin^2\theta \mathrm{d}t\mathrm{d}\varphi + r^2\sin^2\theta\mathrm{d}\varphi^2, & (12)
\end{aligned}$$

where $\Omega$ is a constant. Show that this is actually the Minkowski metric.

As any scalar product, the metric provides a notion of duality between vectors and one-forms (different from the duality between $\boldsymbol{\partial}_\mu$ and $\mathbf{d}x^\mu$). Indeed, given a vector field $\boldsymbol{u}$ one can define the form $\boldsymbol{\eta}_{\boldsymbol{u}} = \boldsymbol{g}(\boldsymbol{u}, \cdot)$, i.e., a map which eats a vector and returns its scalar product with $\boldsymbol{u}$, $\boldsymbol{\eta}_{\boldsymbol{u}} : \boldsymbol{v} \mapsto \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})$. Conversely, to a form $\boldsymbol{\omega}$ we can associate a vector $\boldsymbol{e}^{\boldsymbol{\omega}}$ such that $\boldsymbol{\omega} = \boldsymbol{g}(\boldsymbol{e}^{\boldsymbol{\omega}}, \cdot)$.

**Q3.** If we use the same notation for the components of $\boldsymbol{u}$ and those of its dual form $\boldsymbol{\eta}_{\boldsymbol{u}}$, except for the position of the index, i.e., if we write $\boldsymbol{\eta}_{\boldsymbol{u}} = u_\mu \mathbf{d}x^\mu$, show that the metric can be seen as the *index lowerer*.

**Q4.** Similarly, if we write $\boldsymbol{e}^{\boldsymbol{\omega}} = \omega^\mu \boldsymbol{\partial}_\mu$, show that the *index raiser* is the inverse of the metric.

**Q5.** Simplify the expression $g_{\mu\lambda}g^{\nu\sigma}g^{\lambda\tau}R^\mu{}_{\nu\rho\sigma}A_\tau$, where $R^\mu{}_{\nu\rho\sigma}$, $A_\tau$ are the components of a tensor and a form, respectively.

## 3 The Levi-Civita connection

The Levi-Civita connection is a particular connection associated with the metric. Its coefficients are called the *Christoffel symbols*, and read

$$\Gamma^\mu{}_{\nu\rho} \equiv \frac{1}{2} g^{\mu\sigma} \left(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}\right), \tag{13}$$

where $g^{\mu\sigma}$ are the components of the inverse metric, such that $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$.

**Q1.** Check that $\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}$.

**Q2.** Show that this connection is *metric preserving*, i.e. $\nabla_{\boldsymbol{u}}\boldsymbol{g} = \boldsymbol{0}$ for any vector field $\boldsymbol{u}$.

**Q3.** Deduce from this that the metric can freely get in and out of Levi-Civita covariant derivatives, for example

$$\nabla_\rho \left(g_{\mu\nu}u^\mu v^\nu\right) = g_{\mu\nu}\nabla_\rho(u^\mu v^\nu). \tag{14}$$