Vectors, Forms, and Tensors

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

1 Vectors, forms, and tensors

In this exercise, we review the definitions and properties of vectors, forms, and tensors in differential geometry. Let $\{x^\mu\}$ be a coordinate system over a Lorentzian manifold $\mathcal{M}$. We define the objects $\boldsymbol{\partial}_\mu$ by their action on any function $f : \mathcal{M} \to \mathbb{R}$ by

$$\boldsymbol{\partial}_\mu : f \mapsto \frac{\partial f}{\partial x^\mu}. \tag{1}$$

Q1. Show that, at each event E of $\mathcal{M}$, the set of objects $\boldsymbol{\partial}_\mu$ forms a basis of the tangent space $T_E\mathcal{M}$ of $\mathcal{M}$ at E. You will admit that $\dim T_E\mathcal{M} = \dim \mathcal{M}$ for simplicity.

This means that any vector field on $\mathcal{M}$, $\boldsymbol{X} \in \Gamma(\mathcal{M})$, can be decomposed at every point, over this basis. This defines the *components* of $\boldsymbol{X}$ over the coordinate basis $\{\boldsymbol{\partial}_\mu\}$, with

$$\boldsymbol{X} = X^\mu \boldsymbol{\partial}_\mu. \tag{2}$$

Keep in mind that, rigorously speaking $\boldsymbol{X}, \boldsymbol{\partial}_\mu$ are vectors, while X^μ are simply functions $\mathcal{M} \to \mathbb{R}$. It is however customary to identify a vector with its components, as one would identify linear maps with matrices, and to abusively call the set of X^μ a vector.

Let $\{y^m\}$ be another coordinate system on $\mathcal{M}$, associated with its own set of vectors $\boldsymbol{\partial}_m$ associated with the partial derivatives with respect to y^m, similarly to eq. (1). The vector field $\boldsymbol{X}$ thus also enjoys a decomposition over $\{\boldsymbol{\partial}_m\}$ as

$$\boldsymbol{X} = X^m \boldsymbol{\partial}_m. \tag{3}$$

Q2. What is the relation between $\boldsymbol{\partial}_m$ and $\boldsymbol{\partial}_\mu$? And between the components X^μ and X^m?

A *differential form*, or *one-form*, $\boldsymbol{\omega} \in \Omega^1(\mathcal{M})$, is a field of linear forms. In other words, at each event E of $\mathcal{M}$, $\boldsymbol{\omega}$ eats a vector of the tangent space $T_E\mathcal{M}$, and returns a number. The set of all linear forms at an event E is called the cotangent space $T_E^*\mathcal{M}$ of $\mathcal{M}$ at E. Consider the set of differential forms denoted $\{\mathbf{d}x^\mu\}$, such that

$$\mathbf{d}x^\mu(\boldsymbol{\partial}_\nu) = \delta^\mu_\nu \equiv \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Q3. Show that for any vector field $\boldsymbol{X}$, $\mathbf{d}x^\mu(\boldsymbol{X}) = X^\mu$.

Q4. Justify that, at each $E \in \mathcal{M}$, $\{\mathbf{d}x^\mu\}$ forms a basis of $T_E^*\mathcal{M}$.

Any differential form $\boldsymbol{\omega}$ can thus be decomposed as $\boldsymbol{\omega} = \omega_\mu \mathbf{d}x^\mu$.

Q5. Show that $\omega_\mu = \boldsymbol{\omega}(\boldsymbol{\partial}_\mu)$.

Q6. If another coordinate system $\{y^m\}$ is used, we write $\boldsymbol{\omega} = \omega_m \mathbf{d}y^m$. What is the relation between ω_m and ω_μ? And between $\mathbf{d}y^m$ and $\mathbf{d}x^\mu$?

A *tensor field* $\boldsymbol{T}$ is a composite object, obtained by putting together vectors and forms; it belongs to the tensor product of some copies of $\Gamma(\mathcal{M})$ and $\Omega^1(\mathcal{M})$. For example, if

$$\boldsymbol{T} \in \Gamma(\mathcal{M}) \otimes \Omega^1(\mathcal{M}) \otimes \Omega^1(\mathcal{M}), \tag{5}$$

then $\boldsymbol{T}$ has the properties of a vector combined with two one-forms. In other words, $\boldsymbol{T}$ can be seen as a machine which eats two vectors and returns a vector:

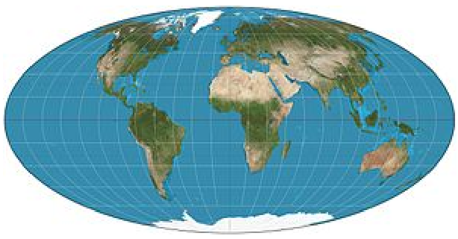
$$\boldsymbol{T} : \begin{array}{l} \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M}) \\ \qquad (\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y}). \end{array} \tag{6}$$

Q7. From the decompositions of vectors and forms, show that the general decomposition of the above tensor is

$$\boldsymbol{T} = T^\mu{}_{\nu\rho}\, \boldsymbol{\partial}_\mu \otimes \mathbf{d}x^\nu \otimes \mathbf{d}x^\rho. \tag{7}$$

Q8. How do the components $T^\mu{}_{\nu\rho}$ transform under a coordinate transformation $\{x^\mu\} \to \{y^m\}$?

2 Mollweide projection of the sphere (continued)



The Mollweide projection is a particular coordinate system which allows one to represent the surface of a sphere on a flat map, where it appears as the interior of an ellipse. Contrary to the Mercator projection (used for most maps of the world), the Mollweide projection better preserves areas but changes angles.

If the sphere has a radius R, a point P on its surface with spherical coordinates $(\theta, \varphi) \in [0, \pi] \times]-\pi, \pi]$ has Mollweide coordinates defined by

$$x = \frac{2\sqrt{2}R}{\pi}\, \varphi \cos\psi \tag{8}$$

$$y = \sqrt{2}R\, \sin\psi, \tag{9}$$

where $\psi \in [-\pi/2, \pi/2]$ is an auxiliary angle implicitly defined by $2\psi + \sin 2\psi = \pi \cos\theta$.

Consider the operator $\boldsymbol{R}$ which rotates any vector tangent to the sphere by an angle δ clockwise. That is the linear map

$$\boldsymbol{R} : \boldsymbol{v} \mapsto \boldsymbol{R}(\boldsymbol{v}) = \boldsymbol{w}, \qquad \text{with} \qquad \begin{bmatrix} w^\theta \\ w^\varphi \end{bmatrix} = \begin{bmatrix} \cos\delta & \sin\theta \sin\delta \\ -\frac{\sin\delta}{\sin\theta} & \cos\delta \end{bmatrix} \begin{bmatrix} v^\theta \\ v^\varphi \end{bmatrix}. \tag{10}$$

The unusual $\sin\theta$ factors in the above rotation matrix are here to correct for the fact that $\boldsymbol{\partial}_\varphi$ is not a unit vector, but has length $\sin\theta$.

Q1. Justify that $\boldsymbol{R}$ is a tensor. What is the relation between its components and the matrix given in eq. (10)?

Q2. Determine the matrix of $\boldsymbol{R}$ in the Mollweide coordinate system.

3 Tensor: Some practice

This is a computational exercise about the notions of vectors, forms and tensors. You can skip it if you feel comfortable enough.
We consider the Cartesian coordinates $x^\mu = (t, x, y, z)$ and the Polar coordinates $\tilde{x}^\mu = (t, r, \phi, z)$. They are related as

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan\left(\frac{y}{x}\right) \end{cases} .$$

We consider the metric

$$\boldsymbol{g} = -\mathbf{d}t^2 + f(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)\,,$$

and the 4-vector

$$\boldsymbol{u} = \gamma\left(\boldsymbol{\partial}_t + \frac{v}{\sqrt{f(t)}}\boldsymbol{\partial}_x\right)\,,$$

where $f(t) > 0$ is a monotonous smooth function, $v < 1$ and $\gamma = (1-v^2)^{-1/2}$.

Q1. Compute u_μ, $g^{\mu\nu}$, $h_{\mu\nu}$, $h_\mu{}^\nu$ and $h^{\mu\nu}$ where the tensor $\boldsymbol{h}$ is defined as

$$\boldsymbol{h} = \boldsymbol{g} + \boldsymbol{u}\otimes\boldsymbol{u}\,,$$

and

$$(\boldsymbol{u}\otimes\boldsymbol{u})_{\mu\nu} = u_\mu u_\nu\,.$$

Q2. Compute $u^2 \equiv u^\mu u^\nu g_{\mu\nu}$ and $h_{\mu\nu}u^\mu$.

We consider the function

$$\alpha(t,x,y,z) = e^{H_0 t}\cos(kx)$$

where H_0 and k have unites of inverse time, and the vector

$$\boldsymbol{r} = t\boldsymbol{\partial}_t + y\boldsymbol{\partial}_y\,.$$

Q3. Compute the form $\omega = \mathbf{d}\alpha$ in the Cartesian basis.

Q4. Compute the scalar product $r^\mu\omega_\mu$ and check that $r^\mu\omega_\mu = \boldsymbol{r}(f)$.

Q5. Compute the components of the metric $\boldsymbol{g}$ and of the 4-vector $\boldsymbol{u}$ in the Polar coordinates.