# The Schwarzschild Solution

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

This sheet deals with the first exact solution of Einstein's equation, discovered by Karl Schwarzschild in December 1915, in the fires of World War I. This solution describes quite accurately the gravitational field created by the Sun, leading to tests of general relativity in the Solar System, but also the most intriguing objects in the Universe: black holes.

The Schwarzschild metric represents the spacetime geometry at the exterior of a static and spherically symmetric body with mass $M$. Its line element reads

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \qquad A(r) \equiv 1 - \frac{r_\mathrm{S}}{r}, \tag{1}$$

where $r_\mathrm{S} \equiv 2GM$ is the Schwarzschild radius of the central object, at $r = 0$. The coordinates $(t, r, \theta, \varphi)$ are sometimes called the Droste coordinates. It is good to keep in mind its order of magnitude for the Sun, namely $r_\mathrm{S}^\odot = 2GM_\odot \approx 3\,\mathrm{km}$.

## 1 Precession of Mercury's perihelion

Like all the planets of the Solar System, Mercury has an elliptic orbit around the Sun. This orbit is not entirely stationary: the axes of the ellipse tend to slowly rotate, with an angular velocity of 5600 arcsec/century. This is known as the precession of Mercury's perihelion. Most of this precession (5026 arcsec/century) is due to the fact that the Sun is not completely spherical, which affects the gravitational field it generates. There is also the effect of the other planets of the Solar System (mostly Venus, Jupiter, and the Earth), responsible for 531 arcsec/century.

Once those effects are taken into account, there are still 43 arcsec/century which are not explained by Newtonian physics. In this exercise, we are going to see that these are due to post-Newtonian corrections of general relativity.

**Q1.** Show that the equation of motion for a freely falling test particle imply

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[A(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] = 0, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) - r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 = 0, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2\theta\,\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right) = 0, \tag{4}$$

where $\tau$ is the particle's proper time.

*HINT*: Start from the length proper time $L = \int\sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu|}d\tau$, and use the Schwarzschild metric to express it in terms of $t(\tau), r(\tau), \theta(\tau), \varphi(\tau)$. Consider small variations of the worldline in just one coordinate at a time, e.g. $t(\tau)$ to $t(\tau) + \delta t(\tau)$ and impose $\delta L = 0$. Carry out this variational procedure for $\theta$ and $\varphi$ as well, and show that the vanishing of $\delta L$ leads to the three differential equations given in the problem.

**Q2.** Conclude that, without loss of generality, we can consider that the motion occurs in the plane $\theta = \pi/2$, and that there are two constants of motion $E \equiv A(r)\dot{t}$, $L \equiv r^2\dot{\varphi}$. Do you understand the fundamental origin of those constants of motion?

**Q3.** Using the normalisation of the four-velocity of the particle and the constants of motion, derive a first-order equation of motion for $r(\tau)$. Introducing the Binet variable $u \equiv 1/r$, show that the trajectory $u(\varphi)$ satisfies the modified second Binet equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{GM}{L^2} + 3GMu^2. \tag{5}$$

In which regime do we recover the Newtonian case?

We are going to solve eq. (5) using a perturbative technique called *multiple-scale expansion*, which is particularly useful to deal with non-linear equations. The underlying motivation is that the two terms of eq. (5) produce changes of $u$ on distinct scales. This is clearer if we introduce the dimensionless quantity $U \equiv u/(GM/L^2) \sim 1$, which yields

$$\frac{\mathrm{d}^2U}{\mathrm{d}\varphi^2} + U = 1 + \varepsilon U^2, \qquad \varepsilon \equiv 3\left(\frac{GM}{L}\right)^2 \tag{6}$$

where we see that while the first term on the right-hand side produces changes of $U$ over angular scales of order unity, the second term produces significant changes on angular scales of order $\varepsilon^{-1} \gg 1$. The multi-scale expansion consists in dealing with $U$ as a function of two independent variables, $U(\varphi_0, \varphi_1)$, where $\varphi_0 = \varphi$ represents the 'fast' evolution and $\varphi_1 = \varepsilon\varphi$ the 'slow' evolution. We also consider a perturbative expansion of $U$ itself, such that

$$U(\varphi) = U_0(\varphi_0, \varphi_1) + \varepsilon U_1(\varphi_0, \varphi_1) + \ldots \tag{7}$$

**Q4.** By consistently expanding eq. (6) at first order in $\varepsilon$, show that

$$\frac{\partial^2 U_0}{\partial {\varphi_0}^2} + U_0 = 1 \tag{8}$$

$$\frac{\partial^2 U_1}{\partial {\varphi_0}^2} + U_1 = U_0^2 - 2\frac{\partial^2 U_0}{\partial\varphi_0\partial\varphi_1}, \tag{9}$$

and give the general solution of eq. (8).

We recognize in eq. (9) the equation for a forced harmonic oscillator, with resonance frequency $\omega_0 = 1$. For the perturbative expansion $U_0 + \varepsilon U_1$ to be meaningful, we have to prevent resonances in $U_1$, by ensuring that the forcing terms with frequency $\omega_0$ on the right-hand side of eq. (9) vanish. Such terms are called *secular*.

**Q5.** Inserting your general solution for $U_0$ in eq. (9), identify the secular terms. Show that the non-resonance condition implies $A - B' = A' + B = 0$ , where $A(\varphi_1)$ and $B(\varphi_1)$ refer to the unknown functions remaining in the general solution for $U_0$.

**Q6.** Solve for $A, B$, and conclude that the lowest order solution for the trajectory of the particle can be written

$$r = \frac{L^2}{GM}\frac{1}{1 + e\cos\left(1 - \frac{3G^2M^2}{L^2}\right)\varphi} \tag{10}$$

**Q7.** Interpret this solution geometrically. Show that at each of its revolution around the Sun, Mercury's perihelion is shifted by an angle

$$\delta = \frac{6\pi GM_\odot}{a(1-e^2)}, \tag{11}$$

where $a$ is the semi-major axis of the Keplerian trajectory. Calculate the shift per century, using that $GM_\odot = 1.5$ km, and the orbital characteristics of Mercury (☿): semi-major axis $a_☿ = 5.8 \times 10^7$ km, eccentricity $e_☿ = 0.2$, and orbital period $T_☿ = 88$ days.