Gravitational Lensing

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

1 The lens equation

Consider a spherically symmetric distribution of mass M generating a weak gravitational field. By solving the null geodesic equation in the corresponding spacetime, you have seen that a light ray emitted and observed very far away from the mass is deflected by an angle

$$\alpha = \frac{4GM}{b}, \tag{1}$$

where b is the impact parameter, which corresponds at lowest order to the minimal distance between the photon and the mass in the trajectory (see fig. 1).

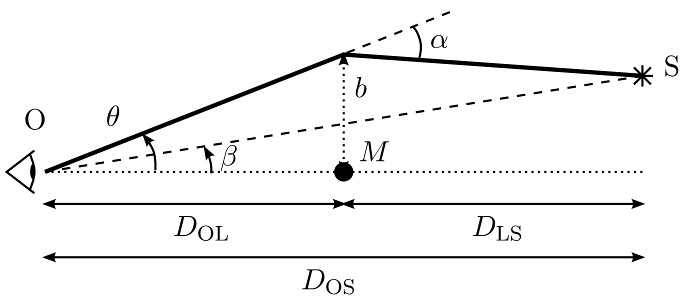

Figure 1 Deflection of light by a spherically symmetric mass M. O, S respectively denote the observer and the source. The angles β, θ are the unlensed and lensed position of the image of S in the observer's celestial sphere, while α is the deflection angle given by eq. (1). Finally, $D_{\mathrm{OL}}, D_{\mathrm{LS}}$ and D_{OS} denote, respectively, the observer-lens, lens-source, and observer-source distances.

Q1. Assuming that $\beta, \theta \ll 1$, show that

$$\beta = \theta - \frac{\theta_{\mathrm{E}}^2}{\theta}. \tag{2}$$

where you will express θ_{E}, called the *Einstein radius*, as a function of $G, M, D_{\mathrm{OL}}, D_{\mathrm{LS}}, D_{\mathrm{OS}}$. Equation (2) is known as the *lens equation*.

Q2. Solve eq. (2) for θ. How many solutions are there? How do you understand this? Why do not we see multiple images of every light source in our daily life?

Q3. The image in fig. 2 has been observed by the Hubble space telescope. Explain this picture. What is the radius of the ring? How can this be used in astrophysics?

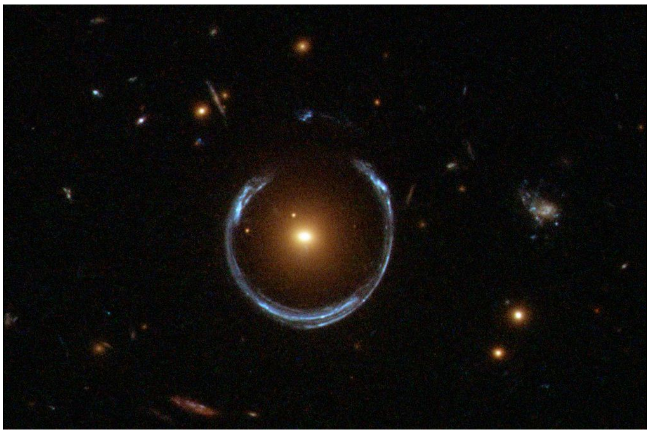

Figure 2 A spectacular gravitational-lensing observation by the Hubble space telescope.

2 Gravitational amplification

Consider a small but non-punctual light source. In this exercise, we are going to show that its apparent luminosity can be amplified by gravitational lensing. A first step consists in slightly generalising the lens equation established in the previous exercise. In the approximation of the small angles, we can consider a fictitious plane in which we represent the angular positions of the sources and images by 2-dimensional vectors. In such conditions, the lens equation is straightforwardly generalised as

$$\boldsymbol{\beta} = \left(1 - \frac{\theta_{\rm E}^2}{\theta^2}\right)\boldsymbol{\theta}, \tag{3}$$

where $\theta \equiv |\boldsymbol{\theta}|$.

The *amplification matrix* $\boldsymbol{\mathcal{A}}$ is defined as the Jacobian matrix of the mapping $\boldsymbol{\theta} \mapsto \boldsymbol{\beta}$, that is

$$\mathcal{A}_{ab} \equiv \frac{\partial \beta_a}{\partial \theta_b}. \tag{4}$$

The altitude of indices a, b does not matter, as $\boldsymbol{\theta}, \boldsymbol{\beta}$ live in a two-dimensional Euclidean space.

Q1. Calculate $\mathcal{A}_{ab}$ explicitly, and show that its determinant reads

$$\det \boldsymbol{\mathcal{A}} = 1 - \left(\frac{\theta_{\rm E}}{\theta}\right)^4. \tag{5}$$

Q2. Justify, from the geometrical meaning of the determinant of a matrix, that if a small (but non-punctual) image is observed in $\boldsymbol{\theta}$, then

$$\mu \equiv \frac{\Omega}{\Omega_0} = \left|\frac{1}{\det \boldsymbol{\mathcal{A}}}\right|, \tag{6}$$

where Ω is the apparent size of the lensed image, and Ω_0 would be its size if there were no lensing. The quantity μ is called the magnification.

It can be shown, although non-trivially, that μ is also the ratio between the observed luminous intensity I and its unlensed counterpart I_0.

Q3. Why is *amplification matrix* a very confusing name for $\boldsymbol{\mathcal{A}}$?

Q4. Show that the magnifications of the two images of a small extended source read

$$\mu_\pm = \left|\frac{1}{2} \pm \frac{u^2+2}{2u\sqrt{u^2+4}}\right|, \tag{7}$$

where $u \equiv \beta/\theta_{\rm E}$. What is the total magnification, taking into account both images?

As an illustration of this phenomenon, the graph of fig. 3 has been obtained by monitoring the luminosity of a star (S) for about three years. Between, roughly, day 400 and day 600, another star (L) passed on the line of sight, producing an enhancement of the apparent luminosity of S.

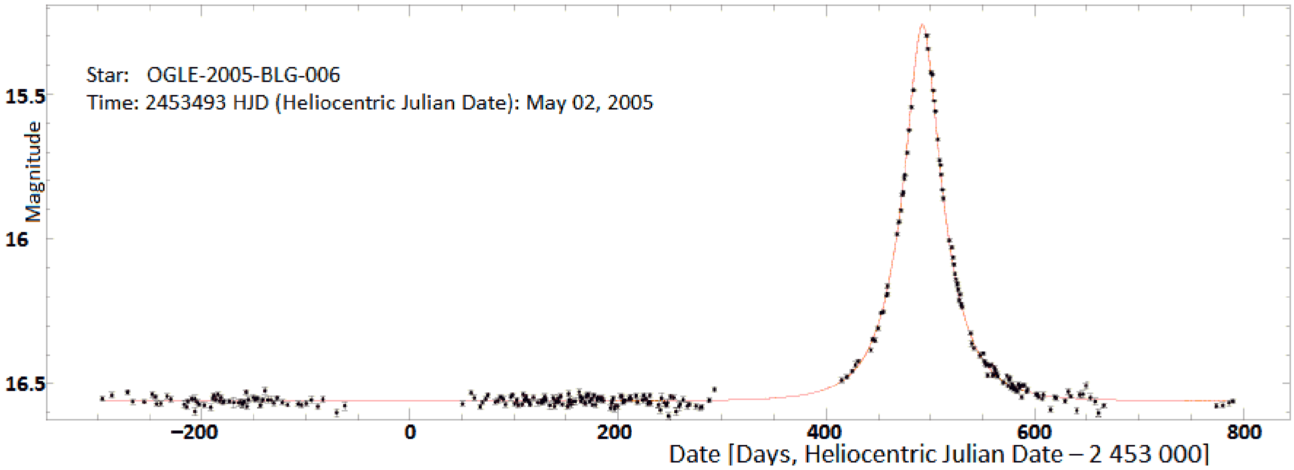

Figure 3 Gravitational amplification of a star by another star acting as a lens.

Q5. Recalling that the magnitude of a star is given by $m = -2.5 \log_{10} I + \text{cst}$, where I is the observed luminous intensity, determine the minimum value of u in this event.