Gravitational Waves

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This exercise is dedicated to gravitational waves, their effect on matter, and their detection. This subject is quite topical: the first detection by LIGO occurred on the 14th of September 2015, followed by six other events from 2015 to 2017. For their decisive contribution to this breakthrough, R. Weiss, K. Thorne, and B. Barish shared the 2017 Nobel Prize in Physics.

1 Tidal field of a gravitational wave

Let us consider a plane monochromatic gravitational wave propagating in the direction $\boldsymbol{\partial}_z$. You have seen in the lectures that the corresponding space-time metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with, in the transverse-traceless gauge (TT),

$$h^{\mathrm{TT}}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}\mathrm{e}^{\mathrm{i}k_\sigma x^\sigma} + \text{c.c.}, \qquad [k_\mu] = [-\omega, 0, 0, \omega] \qquad [\varepsilon_{\mu\nu}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

where c.c. indicates the complex conjugate of the expression, ω is the frequency of the wave, and $h_+, h_\times$ are constant numbers indicating its strain amplitude and polarisation.

Q1. Express the Riemann tensor of this space-time, at linear order, as a function of $h_{\mu\nu,\rho\sigma}$.

Q2. How does $h_{\mu\nu}$ transforms under an infinitesimal coordinate transformation (gauge transformation)? Conclude that the Riemann tensor is gauge invariant.

In order to understand the effect of a gravitational wave on the motion of matter, it is instructive to use Fermi normal coordinates, which can be seen in the present context as a particular gauge. In the vicinity of any time-like geodesic, the metric perturbation reads

$$h^{\mathrm{F}}_{00} = -R_{0i0j}x^i x^j + \mathcal{O}(x^3) \tag{2}$$

$$h^{\mathrm{F}}_{0i} = -\frac{1}{3}(R_{0jik} + R_{0kij})x^j x^k + \mathcal{O}(x^3) \tag{3}$$

$$h^{\mathrm{F}}_{ij} = -\frac{1}{3}(R_{ikj\ell} + R_{i\ell jk})x^k x^\ell + \mathcal{O}(x^3). \tag{4}$$

Q3. Show that, in this gauge, the four-acceleration $\alpha^\mu \equiv u^\nu \nabla_\nu u^\mu$ of a non-relativistic particle reads, at lowest order in its velocity $v^i = \mathrm{d}x^i/\mathrm{d}t$,

$$\alpha^i = \frac{\mathrm{d}v^i}{\mathrm{d}t} - \frac{1}{3}(R_{0jik,0} + R_{0kij,0})x^k x^j + \frac{1}{2}(R_{0k0j}x^k x^j)_{,i}. \tag{5}$$

Q4. Using the fact that the Riemann tensor is gauge invariant, and assuming that $|x| \ll \lambda$, where λ is the wavelength of the gravitational wave, show that we can write

$$\alpha^i = \frac{\mathrm{d}v^i}{\mathrm{d}t} + \partial^i \Phi_{\mathrm{gw}}, \tag{6}$$

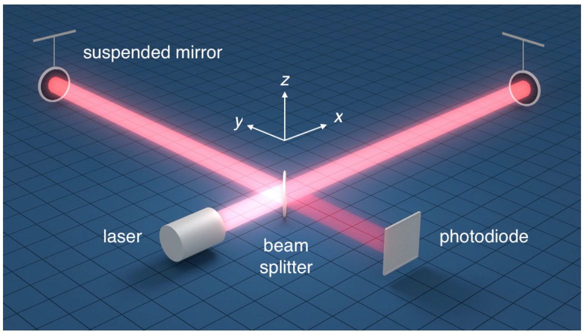
where you will express the pseudo-potential $\Phi_{\rm gw}$ as a function of $h_+, h_\times, \omega$.

Hint: Some terms in eq. (5) will be negligible.

Q5. How do you understand the effect of the gravitational wave on matter?

2 A simplified interferometer

Let us now turn to how LIGO/Virgo actually detect gravitational waves, namely interferometers, as depicted on the right. A laser beam is split into two paths reflected by suspended mirrors. The reflected beams are then recombined on a screen where the interference pattern can be observed. In the following, we set the origin of the coordinate system at the beam splitter, and align the axes x and y with the two arms of the interferometer.



Q6. Give the expression of the luminous intensity I observed on the screen, as a function of the emitted intensity I_0, the lengths X, Y of the arms, and the wavelength of the laser.

Q7. Which default value Δ_0 of the optical path difference $\Delta \equiv 2(X - Y)$ makes the interferometer maximally sensitive to perturbations? In the following, we call $\delta \equiv \Delta - \Delta_0$.

Let us now investigate the effect of a gravitational wave on this interferometer, and more precisely on the suspended mirrors. We have seen in § 1 that a gravitational wave can be seen as an oscillating gravitational pseudo-potential $\Phi_{\rm gw}$. This analysis was performed using Fermi normal coordinates, i.e. in a freely falling frame. Since current interferometers are ground-based (contrary to the future LISA), we first have to move to the Terrestrial frame.

Q8. Propose an approximate coordinate transformation from the coordinates x^μ of a non-relativistic particle in a freely falling frame, to its coordinates X^α in the terrestrial frame. Deduce the relationship between the four-accelerations of the particle in both frames, and conclude that its equation of motion reads

$$\frac{\mathrm{d}V^i}{\mathrm{d}T} = -g^i - \partial^i \Phi_{\rm gw} + \frac{F^i}{m} \, , \tag{7}$$

where $g^i = -g\delta^i_Z$ is the terrestrial gravitational acceleration, F^i is the sum of all non-gravitational forces applied to the particle, and m is mass.

Q9. Show that, for small perturbations of the positions of the mirror compared to their equilibrium state, we have

$$\ddot{\delta} + \omega_0^2 \delta = 2\omega^2 h_+ L_0 \cos \omega t, \tag{8}$$

where you will specify the expression of ω_0, L_0.

Q10. What is the resulting amplitude of the oscillations of δ? Given that the height of LIGO's suspensions is on the order of 1 m, and that the typical frequency of the observed gravitational waves is 100 Hz, in which regime is the oscillator?

Q11. Given that the frequency of the laser of LIGO is $\lambda_{\rm l} = 1\,\mu\mathrm{m}$, that the effective[1] length of the arms is $L_0 = 1.2 \times 10^6$ m, and that the strain amplitude of the first wave detected was $h_+ = 10^{-21}$, deduce the sensitivity of the LIGO instrument with respect to changes of luminous intensity.

[1]LIGO contains a resonant cavity that essentially multiplies the 4 km-arms by a factor 300.

Q12. Gravitational waves are often presented as “ripples in spacetime”, which “stretch space as they propagate”. What do you think about these expressions? Could we detect gravitational waves if the mirrors were standing on the ground?