Prelude to Vectors, Forms, and Tensors

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1 Electromagnetic field

Note. In this exercise, we will consider *inertial* coordinates only, for which the metric takes the standard Minkowski form $[\eta_{\mu\nu}] = [\eta^{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$. You will assume that this metric can be used to raise and lower indices, in the sense that $A^\mu = \eta^{\mu\nu} A_\nu$ and $A_\mu = \eta_{\mu\nu} A^\nu$. Do not worry if this seems obscure, it will be extensively discussed later in the course.

In relativistic electrodynamics, the fundamental field is a one-form $\boldsymbol{A} = A_\mu \mathbf{d}x^\mu$ associated with the scalar and vector potentials of electromagnetism, as $(A_\mu) = (-V, \vec{A})$. We then define the following quantity:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]}. \tag{1}$$

Q1. How does $F_{\mu\nu}$ transform under a general coordinate transformation $\{x^\mu\} \to \{y^\alpha\}$. What is the geometric status of this quantity?

Recall that, in standard electromagnetism, the relations between the electromagnetic field and the electromagnetic potentials are

$$E_i = -\partial_i V - \partial_t A_i, \tag{2}$$

$$B^i = \varepsilon^{ijk} \partial_j A_k, \tag{3}$$

where ε^{ijk} is the three-dimensional permutation symbol.

Q2. Express the components E_i and B^i of the electromagnetic field as a function of $F_{\mu\nu}$. Extending the components of the electric and magnetic fields to four dimensions, assuming $E^0 = B^0 = 0$, show that

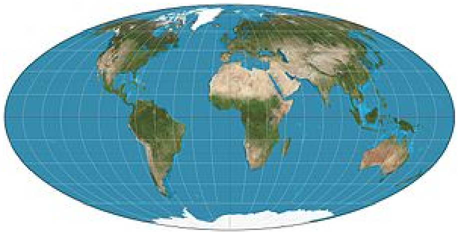
$$E^\mu = F^{\mu\nu} u_\nu, \tag{4}$$

$$B^\mu = -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}, \tag{5}$$

where $\boldsymbol{u}$ is the four-velocity of an observer at rest with respect to the coordinate system (t, x^i). The symbol $\varepsilon_{\mu\nu\rho\sigma}$ denotes the four-dimensional permutation symbol, with the convention $\varepsilon^{0ijk} = -\varepsilon^{ijk}$. What is the advantage of eqs. (4), (5) over eqs. (2), (3)?

Q3. Suppose that another observer, with a different four-velocity $\boldsymbol{v} \neq \boldsymbol{u}$, performs measurements of this electromagnetic field. Does she obtain the same electric and magnetic fields?

2 Mollweide projection of the sphere



The Mollweide projection is a particular coordinate system which allows one to represent the surface of a sphere on a flat map, where it appears as the interior of an ellipse. Contrary to the Mercator projection (used for most maps of the world), the Mollweide projection better preserves areas but changes angles.

If the sphere has a radius R, a point P on its surface with spherical coordinates $(\theta, \varphi) \in [0, \pi] \times [-\pi, \pi]$ has Mollweide coordinates defined by

$$x = \frac{2\sqrt{2}R}{\pi} \, \varphi \cos\psi \tag{6}$$

$$y = \sqrt{2}R \, \sin\psi, \tag{7}$$

where $\psi \in [-\pi/2, \pi/2]$ is an auxiliary angle implicitly defined by $2\psi + \sin 2\psi = \pi\cos\theta$.

Q1. What are the Mollweide coordinates of the poles?

Q2. Show that the Mollweide components of a vector field $\boldsymbol{v} = v^\theta \hat{\boldsymbol{\theta}} + v^\varphi \hat{\boldsymbol{\varphi}}$ read

$$v^x = -\frac{xyF(y)}{\sqrt{2R^2 - y^2}} \, v^\theta + \frac{2}{\pi}\sqrt{2R^2 - y^2} \, v^\varphi, \tag{8}$$

$$v^y = \sqrt{2R^2 - y^2} \, F(y) \, v^\theta, \tag{9}$$

with

$$F(y) \equiv -\frac{\pi}{4}\left(1 - \frac{y^2}{2R^2}\right)^{-1} \sqrt{1 - \frac{4}{\pi^2}\left[\arcsin\left(\frac{y}{\sqrt{2}R}\right) + \frac{y}{\sqrt{2}R}\sqrt{1 - \frac{y^2}{2R^2}}\right]^2} \tag{10}$$

Q3. Consider winds whose velocity pattern on the globe reads $\boldsymbol{v} = v_0 \hat{\boldsymbol{\varphi}}$, where v_0 is a constant. How does it appear on the Mollweide projection?