Special Relativity

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In this series of exercises, and for all the next ones, we apply Einstein's convention of summation over repeated indices.

1 The twin paradox reloaded

Consider twin sisters, Alice and Brienne, who have lived together for twenty years. For her 20th birthday, Brienne is offered a space journey around the Sun. The purpose of this exercise it to calculate the age of both Alice and Brienne when the latter returns from her trip. We assume everything to happen in the Minkowski spacetime, whose metric, in an inertial coordinate system $(x^\mu) = (ct, x, y, z)$, reads

$$\mathrm{d}s^2 \equiv \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1}$$

where Greek indices run from 0 to 3, while Latin indices run from 1 to 3, and δ_{ij} is the Krönecker symbol. We assume that Alice remains at rest on Earth, assumed to be at the origin of this coordinate system ($x^i_A = 0$), while Brienne follows a circular trajectory around the Sun, whose centre is located at $(R, 0, 0)$, with

$$x_\mathrm{B} = R - R\cos(vt/R) \tag{2}$$
$$y_\mathrm{B} = R\sin(vt/R) \tag{3}$$
$$z_\mathrm{B} = 0, \tag{4}$$

and where v is the velocity of Brienne's rocket. The event of her departure has coordinates $x^\mu = 0$.

Q1. What are the coordinates of the event corresponding to Brienne's arrival back on Earth? Sketch the *worldlines* of both Alice and Brienne, on a 3-dimensional diagram where time t is the vertical axis, while x, y span horizontal planes.

Q2. Suppose that the coordinates of an observer change by the infinitesimal amount $\mathrm{d}x^\mu$. At which condition is this displacement said to be *timelike*? In that case, what is the *proper duration* $\mathrm{d}\tau$, as measured by the observer's clock, for this displacement?

Q3. When the time coordinate changes by an amount $\mathrm{d}t$, what is the proper time elapsed for Alice ? For Brienne ? Calculate the duration of Brienne's trip as measured by Alice, $\Delta\tau_\mathrm{A}$, and Brienne, $\Delta\tau_\mathrm{B}$. Which one is the shortest? Compute the relative difference $\Delta\tau_\mathrm{A}/\Delta\tau_\mathrm{B} - 1$ for $v = 0.1c$.

Now suppose that Brienne has a more complicated trajectory $x^i(t)$; say she swings by Mars and Saturn, stops a bit on the asteroid belt, and finally comes back.

Q4. What is the general expression for the proper duration $\Delta\tau_\mathrm{B}$ of Brienne's trip as measured by her own clock? Does she *always* come back younger or older than her sister?

2 A dancing particle

Let $\eta_{\mu\nu} = (-1, +1, +1, +1)$ be the Minkowski metric. Let us consider a particle whose coordinates (t, x, y, z) are given by

$$t = f(\lambda)$$
$$x = R\cos(\omega\lambda)$$
$$y = R\sin(\omega\lambda)$$
$$z = c\lambda\,,$$

where λ is a parameter with dimensions of time such that $f(0) = 0$ and $f'(\lambda) > 0$, and ω is a frequency with dimensions of inverse time. We define the position 4-vector of the particle as[1]

$$x^\mu = (ct, x, y, z)\,.$$

Q1. Compute the 4-velocity of the particle given by

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\,.$$

Q2. Compute its covariant form $u_\mu \equiv \eta_{\mu\nu}u^\nu$.

Q3. Compute the squared norm of the 4-velocity given by $u^2 \equiv u^\mu u_\mu$.

Q4. Determine the function $f(\lambda)$ assuming that $u^2 = -c^2$.

Q5. Sketch the trajectory of the particle in a 3D spatial plot. Is the particle massive or massless?

[1] Note that there is slight notation abuse as we use x for both the first spatial component and the 4-vector, but there should be no confusion as the 4-vector is always denoted with an index.