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# QUANTUM PHYSICS III

## Solutions to Problem Set 8

4 November 2025

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### 1. On integrals involving the delta-function

1. Consider the integral

$$I = \int_{-\infty}^{\infty} dx f(x) \delta(ax^2 + bx + c). \quad (1)$$

Denote the argument of the delta-function by  $g(x)$ . There are several possibilities :

- If the equation  $g(x) = 0$  has no real roots, then the argument of the delta-function is never zero, hence  $I = 0$ .
- Suppose that the equation  $g(x) = 0$  has two different real roots  $x_{1,2}$ . Near each of them the function  $g(x)$  can be written as  $g(x) = g'(x_{1,2})(x - x_{1,2}) + \mathcal{O}((x - x_{1,2})^2)$ . Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be small neighborhoods of the points  $x_1$  and  $x_2$  correspondingly. The integral  $I$  becomes

$$I = \int_{\mathcal{O}_1} \frac{dy}{|g'(x_1 + y)|} f(x_1 + y) \delta(y) + \int_{\mathcal{O}_2} \frac{dy}{|g'(x_2 + y)|} f(x_2 + y) \delta(y), \quad (2)$$

where we made the change of variable  $y = x - x_1$  in the first integral,  $y = x - x_2$  in the second integral, and used the property of the delta-function

$$\delta(\alpha x) = \frac{1}{|\alpha|} \delta(x), \quad (3)$$

with  $\alpha$  some constant. Taking the integrals, we have

$$I = \frac{f(x_1)}{|g'(x_1)|} + \frac{f(x_2)}{|g'(x_2)|}. \quad (4)$$

Finally,  $|g'(x_1)| = |g'(x_2)| = |a(x_1 - x_2)| = \sqrt{b^2 - 4ac}$ , and

$$I = (f(x_1) + f(x_2))(b^2 - 4ac)^{-1/2}. \quad (5)$$

- Suppose now that  $x_1 = x_2 = x_0$ . Expanding  $g(x)$  around  $x_0$  and changing the variable  $y = x - x_0$ , we arrive at

$$I = \int_{-\infty}^{\infty} \frac{dy f(x_0 + y)}{|g'(x_0 + y)|} \delta(y) = \lim_{x \rightarrow x_0} \frac{f(x)}{2|a(x - x_0)|} = \begin{cases} \infty, & f(x_0) \neq 0, \\ \frac{f'(x_0)}{2|a|}, & f(x_0) = 0. \end{cases} \quad (6)$$

2. Recall that

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad (7)$$

where  $i$  numerates the roots of the function  $f$ . In our case  $E_p = \frac{p^2}{2m}$ , and

$$\begin{aligned} \int d^3 \mathbf{p} \delta(E_{p'} - E_p) f(\mathbf{p}) &= \int d\Omega dp p^2 \delta(E_p - E_{p'}) f(\mathbf{p}) \\ &= \int d\Omega dp p^2 \delta(p - p') \frac{2m}{2p'} f(\mathbf{p}) \\ &= mp' \int d\Omega f(\mathbf{n}), \end{aligned} \quad (8)$$

where

$$\mathbf{n} = |\mathbf{p}'| \frac{\mathbf{p}}{|\mathbf{p}|} \quad (9)$$

is a vector of modulus  $|\mathbf{p}'|$  in the direction of  $\mathbf{p}$ .

## 2. Free particle's Green function in three dimensions

1. By definition,

$$\hat{G}_0(z) = \frac{1}{z - \hat{H}_0}. \quad (10)$$

This means that

$$\hat{G}_0(z)|\mathbf{p}\rangle = \frac{1}{z - E_p} |\mathbf{p}\rangle, \quad E_p = \frac{p^2}{2m}. \quad (11)$$

Therefore,

$$\langle \mathbf{x} | \hat{G}_0(z) | \mathbf{x}' \rangle = \int d^3 \mathbf{p} \langle \mathbf{x} | \hat{G}_0(z) | \mathbf{p} \rangle \langle \mathbf{p} | \mathbf{x}' \rangle = \frac{1}{(2\pi)^3} \int d^3 \mathbf{p} \frac{e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')}}{z - E_p}. \quad (12)$$

2. Let us first compute the angular part of the integral :

$$\begin{aligned} \langle \mathbf{x} | \hat{G}_0(z) | \mathbf{x}' \rangle &= 2\pi \frac{1}{(2\pi)^3} \int_0^\infty dp p^2 \int_0^\pi d\theta \sin \theta \frac{e^{ip|\mathbf{x} - \mathbf{x}'| \cos \theta}}{z - E_p} \\ &= -\frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{ip|\mathbf{x} - \mathbf{x}'|} \frac{1}{z - E_p} (e^{-ip|\mathbf{x} - \mathbf{x}'|} - e^{ip|\mathbf{x} - \mathbf{x}'|}) \\ &= -\frac{im}{2\pi^2 |\mathbf{x} - \mathbf{x}'|} \int_{-\infty}^\infty dp p \frac{e^{ip|\mathbf{x} - \mathbf{x}'|}}{2mz - p^2}. \end{aligned} \quad (13)$$

The resulting integral can be computed by the method of residues. To this end, we close the contour of integration in the plane of complex  $p$  as shown in figure 1. This does not change the value of the integral, since in the upper half-plane the integrand approaches zero exponentially fast when the radius of the semi-circle goes to infinity. The integrand has two poles at  $p = \pm \sqrt{2mz}$ . Recall that  $z = E + i\epsilon$ ,  $\epsilon > 0$ , hence the pole contributing to the integral is the one at  $p = +\sqrt{2mz}$ . Thus,

$$\begin{aligned}\langle \mathbf{x} | \hat{G}_0(z) | \mathbf{x}' \rangle &= -\frac{im}{2\pi^2 |\mathbf{x} - \mathbf{x}'|} 2\pi i \operatorname{res}_{p=\sqrt{2mz}} \frac{pe^{ip|\mathbf{x}-\mathbf{x}'|}}{(p - \sqrt{2mz})(p + \sqrt{2mz})} \\ &= \frac{m}{2\pi} \frac{e^{i\sqrt{2mz}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}.\end{aligned}\quad (14)$$

3. The formula (12) tells us that the Fourier transform of the function  $G_0(z, \mathbf{x}, \mathbf{x}') = \langle \mathbf{x} | \hat{G}_0(z) | \mathbf{x}' \rangle$  is

$$G_0(z, \mathbf{p}, \mathbf{p}') = \delta(\mathbf{p} - \mathbf{p}') \frac{1}{z - E_p}.\quad (15)$$

This implies in particular the conservation of the free particle momentum. We now use the momentum representation of the Green function to yield

$$\begin{aligned}\langle \mathbf{x} | (z - \hat{H}_0) \hat{G}_0(z) | \mathbf{x}' \rangle &= \int d^3 \mathbf{p} d^3 \mathbf{p}' d^3 \mathbf{p}'' \langle \mathbf{x} | \mathbf{p} \rangle \langle \mathbf{p} | z - \hat{H}_0 | \mathbf{p}'' \rangle \langle \mathbf{p}'' | \hat{G}_0(z) | \mathbf{p}' \rangle \langle \mathbf{p}' | \mathbf{x}' \rangle \\ &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{p} d^3 \mathbf{p}' d^3 \mathbf{p}'' e^{i\mathbf{p}\cdot\mathbf{x}} \delta(\mathbf{p} - \mathbf{p}'') (z - E_{p''}) \delta(\mathbf{p}'' - \mathbf{p}') \frac{1}{z - E_{p'}} e^{-i\mathbf{p}'\cdot\mathbf{x}'} \\ &= \frac{1}{(2\pi)^{3/2}} \int d^3 \mathbf{p} d^3 \mathbf{p}' e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{p}'\cdot\mathbf{x}'} \delta(\mathbf{p} - \mathbf{p}') \\ &= \frac{1}{(2\pi)^{3/2}} \int d^3 \mathbf{p} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} = \delta(\mathbf{x} - \mathbf{x}').\end{aligned}\quad (16)$$

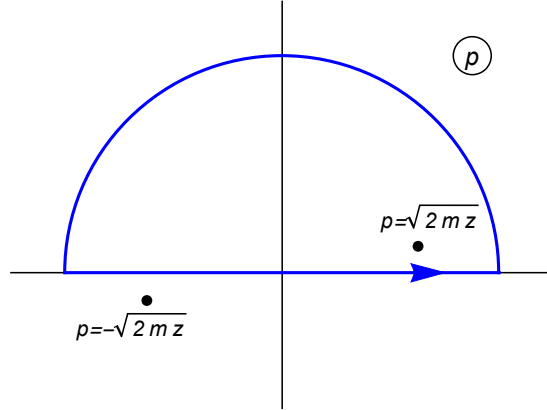


FIG. 1 – The contour of integration

4. The matrix element  $G_0(z, \mathbf{x}, \mathbf{x}')$  as a function of the complex variable  $z$  has a branch cut along the real positive values of  $z$ . To calculate the difference between the points on the opposite sides of the branch cut, one should continue analytically the function  $\sqrt{z}$  from the one side to another. This gives,

$$\begin{aligned}G_0(E + i\epsilon, \mathbf{x}, \mathbf{x}') - G_0(E - i\epsilon, \mathbf{x}, \mathbf{x}') &= -\frac{m}{2\pi |\mathbf{x} - \mathbf{x}'|} \left( e^{i\sqrt{2mE}|\mathbf{x}-\mathbf{x}'|} - e^{-i\sqrt{2mE}|\mathbf{x}-\mathbf{x}'|} \right) \\ &= -\frac{im}{\pi |\mathbf{x} - \mathbf{x}'|} \sin \left( \sqrt{2mE} |\mathbf{x} - \mathbf{x}'| \right).\end{aligned}\quad (17)$$

5. For large values of  $x = |\mathbf{x}|$ , the modulus  $|\mathbf{x} - \mathbf{x}'|$  can be expanded as

$$|\mathbf{x} - \mathbf{x}'| = x \sqrt{1 - \frac{2\mathbf{x} \cdot \mathbf{x}'}{x^2} + \frac{x'^2}{x^2}} \approx x - \frac{\mathbf{x} \cdot \mathbf{x}'}{x}. \quad (18)$$

Thus,

$$G_0(z, \mathbf{x}, \mathbf{x}') \approx -\frac{m}{2\pi} \frac{\exp\left[i\sqrt{2mz}\left(x - \frac{\mathbf{x} \cdot \mathbf{x}'}{x}\right)\right]}{x}, \quad x \rightarrow \infty. \quad (19)$$

### 3. Friedel sum rule

1. The eigenstates of the Hamiltonian  $\hat{H}$  form an orthonormal basis of states. This fact allows us to write

$$\hat{G}(x + i\epsilon) = \frac{1}{x - \hat{H} + i\epsilon} = \sum_n \frac{|n\rangle\langle n|}{x - E_n + i\epsilon}. \quad (20)$$

Here by  $|n\rangle$  we understand both the bound states and the scattering states. For the matrix element we have,

$$G_{nm}(x + i\epsilon) = \langle n|\hat{G}(x + i\epsilon)|m\rangle = \sum_n \frac{\delta_{nm}}{x - E_n + i\epsilon}, \quad (21)$$

or

$$G_{mm}(x + i\epsilon) = \frac{1}{x - E_n + i\epsilon} = \frac{d}{dx} \log G_{mm}^{-1}(x + i\epsilon) = -\frac{d}{dx} \log G_{mm}(x + i\epsilon). \quad (22)$$

Now we turn to the function  $N(x)$ . Using the relation

$$\frac{1}{x + i\epsilon} = -i\pi\delta(x) + \mathcal{P}\frac{1}{x}, \quad (23)$$

we have,

$$N(x) = \sum_n \delta(x - E_n) = -\frac{1}{\pi} \sum_n \text{Im} \frac{1}{x - E_n + i\epsilon} = -\frac{1}{\pi} \text{Im} \sum_n G_{mm}(x + i\epsilon). \quad (24)$$

Substitution of the expression (22) then leads to

$$N(x) = \frac{1}{\pi} \text{Im} \sum_n \frac{d}{dx} \log G_{mm}(x + i\epsilon) = \frac{1}{\pi} \frac{d}{dx} \text{Im} \log \det \hat{G}(x + i\epsilon). \quad (25)$$

2. From eq. (25) it follows that

$$\begin{aligned} N(x) - N_0(x) &= \frac{1}{\pi} \frac{d}{dx} \text{Im} \left[ \log \det \hat{G}(x + i\epsilon) - \log \det \hat{G}_0(x + i\epsilon) \right] \\ &= \frac{1}{\pi} \frac{d}{dx} \text{Im} \log \det \hat{G} \hat{G}_0^{-1}(x + i\epsilon). \end{aligned} \quad (26)$$

Writing  $\det \hat{G} \hat{G}_0^{-1}$  as  $|\det \hat{G} \hat{G}_0^{-1}| e^{i \arg \det \hat{G} \hat{G}_0^{-1}}$  gives

$$N(x) - N_0(x) = \frac{1}{\pi} \frac{d}{dx} \arg \det \hat{G} \hat{G}_0^{-1}(x + i\epsilon). \quad (27)$$

#### 4. Slow scattering in a gas

The problem concerns atom-atom scattering inside a gas. The condition

$$p \lesssim \frac{\hbar}{R} \quad (28)$$

implies  $\mu v_r R \lesssim \hbar$ , where  $\mu = \frac{1}{2}m_p$  is the reduced mass of the two atoms,  $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2$  is the relative velocity between the two atoms of velocities  $\mathbf{v}_1, \mathbf{v}_2$ , and  $R = 4 \text{ \AA}$ . In thermal equilibrium

$$\frac{1}{2}m_p \langle v^2 \rangle = \frac{3}{2}kT, \quad (29)$$

with  $k$  the Boltzmann constant and  $T$  the temperature. The mean-square value of the relative speed  $v_r$  is

$$\langle v_r^2 \rangle = \langle (\mathbf{v}_1 - \mathbf{v}_2)^2 \rangle = \langle v_1^2 + v_2^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) \rangle = 2\langle v^2 \rangle = \frac{6kT}{m_p}. \quad (30)$$

Thus,

$$\mu R v_r \approx \frac{m_p R}{2} \sqrt{\frac{6kT}{m_p}} \lesssim \hbar, \quad (31)$$

i.e.,

$$T \lesssim \frac{2\hbar^2}{3m_p c^2} \left(\frac{c}{R}\right)^2 \frac{1}{k} = 2 \text{ K}. \quad (32)$$