QUANTUM PHYSICS III

Solutions to Problem Set 7 28 October 2025

1. Interaction picture

1. Recalling the relation between states and operators in the Schroedinger and Heisenberg pictures, we have

$$\begin{aligned} \Psi_I(t) &= \hat{U}_0^\dagger(t)\Psi_S(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\Psi_H \ , \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t)\hat{A}_S\hat{U}_0(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\hat{A}_H(t)\hat{U}^\dagger(t)\hat{U}_0(t) \ . \end{aligned} \tag{1}$$

2. The evolution equation for the wave function in the interaction picture is obtained straightforwardly :

$$\begin{aligned} -\frac{\hbar}{i}\frac{d}{dt}\Psi_I(t) &= -\frac{\hbar}{i}\frac{d}{dt}\hat{U}_0^\dagger(t)\Psi_S(t) = -\frac{\hbar}{i}\frac{d\hat{U}_0^\dagger(t)}{dt}\Psi_S(t) - \frac{\hbar}{i}\hat{U}_0^\dagger(t)\frac{d\Psi_S(t)}{dt} \\ &= -\hat{U}_0^\dagger(t)\hat{H}_0\Psi_S(t) + \hat{U}_0^\dagger(t)(\hat{H}_0 + \hat{V})\Psi_S(t) \\ &= \hat{U}_0^\dagger(t)\hat{V}\hat{U}_0(t)\Psi_I(t) = \hat{V}_I(t)\Psi_I(t) \ , \end{aligned} \tag{2}$$

where in the last line we used the fact that $\Psi_S(t) = \hat{U}_0(t)\Psi_I(t)$.

3. Similarly to the Schroedinger picture in which $\Psi_S(t) = \hat{U}(t)\Psi(0)$, one can define an operator $\hat{U}_I(t)$ such that $\Psi_I(t) = \hat{U}_I(t)\Psi(0)$. From eq. (1) we have

$$\Psi_I(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\Psi(0) \ . \tag{3}$$

Hence $\hat{U}_I(t) = \hat{U}_0^\dagger(t)\hat{U}(t)$. Substitution of eq. (3) into eq. (2) gives

$$-\frac{\hbar}{i}\frac{d\hat{U}_I(t)}{dt} = \hat{V}_I(t)\hat{U}_I(t) \ . \tag{4}$$

The initial condition for the operator $\hat{U}_I(t)$ is $\hat{U}_I(0) = 1$.

2. Unitarity versus isometry

1. (a) From $\mathcal{D}(\hat{U}) = \mathcal{H}$ and $\mathcal{R}(\hat{U}) = \mathcal{H}$ it follows that there is an inverse operator $\hat{U}^{-1}$ such that $\hat{U}\hat{U}^{-1} = 1$. Then, from $\hat{U}^\dagger\hat{U}\hat{U}^{-1} = \hat{U}^{-1}$ it follows that $\hat{U}^\dagger = \hat{U}^{-1}$. Therefore, $\hat{U}^{-1}\hat{U}\hat{U}^\dagger = \hat{U}^{-1}$, and $\hat{U}\hat{U}^\dagger = 1$.
(b) From $\hat{U}^\dagger\hat{U} = 1$ if follows that $\mathcal{R}(\hat{U}) \subseteq \mathcal{D}(\hat{U}^\dagger) = \mathcal{H}$. Then, from $\hat{U}\hat{U}^\dagger = 1$ it follows that for any element x from $\mathcal{D}(\hat{U}^\dagger)$ the operator $\hat{U}$ must map back to x the image of x under the action of $\hat{U}^\dagger$. Hence, $\mathcal{R}(\hat{U}) = \mathcal{D}(\hat{U}^\dagger) = \mathcal{H}$.

2. One should prove that if $\mathcal{H}$ is finite-dimensional, then $\mathcal{R}(\hat{U}) = \mathcal{H}$ follows from $\mathcal{D}(\hat{U}) = \mathcal{H}$. After that, $\hat{U}^\dagger \hat{U} = 1$ will follow from $\hat{U}\hat{U}^\dagger = 1$. To prove the coincidence of the domain and the range of $\hat{U}$, we enumerate the basis in $\mathcal{H}$ as $|1\rangle, ..., |n\rangle$. Then, $\hat{U}$ is represented by an $n \times n$ matrix. Since $\hat{U}^\dagger \hat{U} = 1$, it follows that $\det \hat{U} = 1$. Hence, $\hat{U}$ is non-degenerate and there is an inverse $n \times n$ matrix $\hat{U}^{-1}$. Thus, $\mathcal{D}(\hat{U}^{-1}) = \mathcal{H}$ and $\mathcal{R}(\hat{U}) = \mathcal{H}$.

3. To construct the required sequence, one can use the Gram–Schmidt orthogonalization process. Select the basis $|1\rangle, ..., |n\rangle, ...$ in $\mathcal{H}$. Choose the action of $\hat{U}(\lambda)$ on the vector $|1\rangle$ as follows,

$$\hat{U}(\lambda)|1\rangle = |1'\rangle = \sqrt{\lambda}\,|1\rangle + \sqrt{1-\lambda}\,|2\rangle\,. \tag{5}$$

We will consider λ in the range $[0, 1]$. It is clear that $\langle 1'|1'\rangle = 1$. Now define the action of $\hat{U}(\lambda)$ on $|2\rangle$ as $\hat{U}(\lambda)|2\rangle = |2'\rangle = c_{21}|1\rangle + c_{22}|2\rangle + c_{23}|3\rangle$, and choose the coefficients c_{21}, c_{22}, c_{23} such that $\langle 1'|2'\rangle = 0$ and $\langle 2'|2'\rangle = 1$. The orthogonality condition fixes the values of c_{21} and c_{22},

$$c_{21} = -f(\lambda)\sqrt{1-\lambda}\,, \quad c_{22} = f(\lambda)\sqrt{\lambda}\,, \tag{6}$$

up to some arbitrary function $f(\lambda)$. One can choose, for example, $f(\lambda) = \sqrt{\lambda}$. Then, c_{23} is fixed by the normalization condition,

$$c_{23} = \sqrt{1-\lambda^2-\lambda(1-\lambda)}\,. \tag{7}$$

Hence

$$\hat{U}(\lambda)|2\rangle = |2'\rangle = -\sqrt{\lambda}\sqrt{1-\lambda}\,|1\rangle + \lambda\,|2\rangle + \sqrt{1-\lambda^2-\lambda(1-\lambda)}\,|3\rangle\,. \tag{8}$$

The next step of this procedure gives,

$$\hat{U}(\lambda)|3\rangle = |3'\rangle = c_{31}|1\rangle + c_{32}|2\rangle + c_{33}|3\rangle + c_{34}|4\rangle\,, \tag{9}$$

where

$$\begin{aligned} c_{31} &= \sqrt{\lambda}(\lambda - \sqrt{1-\lambda^2-\lambda(1-\lambda)})\,, \\ c_{32} &= \sqrt{\lambda}(\sqrt{1-\lambda^2-\lambda(1-\lambda)} + \sqrt{\lambda}\sqrt{1-\lambda})\,, \\ c_{33} &= \sqrt{\lambda}(-\sqrt{\lambda}\sqrt{1-\lambda} - \lambda)\,, \\ c_{34} &= \sqrt{1-c_{31}-c_{32}-c_{33}}\,. \end{aligned} \tag{10}$$

Since $\mathcal{H}$ is infinte-dimensional, one can continue this process and define the action of $\hat{U}(\lambda)$ on arbitrary $|n\rangle$. For all $\lambda \in (0, 1]$, the operator $\hat{U}(\lambda)$ is unitary by construction. However, it is easy to see that in the limit of zero λ it becomes a "shift" operator

$$\hat{U}(0)|i\rangle \equiv \hat{\Omega}|i\rangle = |i+1\rangle, \quad \forall i, \tag{11}$$

whose range does not include the vector $|1\rangle$.

3. Semiclassical S-matrix in one dimension

We want to compute the matrix element

$$S(p,\sigma,p',\sigma') = \int dxdy \langle \psi_{p'\sigma'}|x\rangle \langle x|S|y\rangle \langle y|\psi_{p,\sigma}\rangle \tag{12}$$

By inserting a complete set of momentum states, we also know

$$\langle x|S|y\rangle = \int dqdq' \langle x|q'\rangle \langle q'|S|q\rangle \langle q|y\rangle \tag{13}$$

Now, the question asks us to consider ψ_{out} just to be the transmitted wave. This tells us

$$\langle q'|S|q\rangle = D(q)\delta(q-q') \tag{14}$$

The delta function ensures that the wave is transmitted (were we to consider reflection, there would be an additional contribution).
Thus we need to compute :

$$S(p,\sigma,p',\sigma') = \int dxdydq\psi_{p'\sigma'}(x,t)\psi^*_{p,\sigma}(y,t)D(q)e^{iq(x-y)} \tag{15}$$

Now we plug in $D(q) = 1 - e^{-q^2/q_0^2}$. The first part gives 1, as the wave packets are normalised. It then remains to compute the following Gaussian integrals :

$$\int dxdydqe^{-(x-\frac{p}{m}t)^2/(4\sigma^2)}e^{-(y-\frac{p'}{m}t)^2/(4\sigma'^2)}e^{-q^2/p_o^2}e^{iq(x-y)} = \frac{4\pi^{3/2}q_0}{\sqrt{\frac{1+q_0^2(\sigma^2+\sigma'^2)}{\sigma^2\sigma'^2}}}e^{-\frac{(p-p')^2q_0^2t^2}{4m^2(1+q_0^2(\sigma^2+\sigma'^2))}} \tag{16}$$

These integrals are done by successively completing the squares.