QUANTUM PHYSICS III

Solutions to Problem Set 5 7 October 2025

1. Perturbation of potential and WKB

1. From the quantization conditions

$$\oint p_0 dx = 2\pi\hbar\left(n+\frac{1}{2}\right)\,,\quad \oint p dx = 2\pi\hbar\left(n+\frac{1}{2}\right) \tag{1}$$

it follows that

$$\oint (p-p_0)\,dx = 0\,, \tag{2}$$

where we used the fact that the change of the turning points under the small perturbation $\delta V(x)$ can be neglected to the first order in $\delta V(x)/V(x)$. Next,

$$p = \sqrt{2m(E-V)\left(1+\frac{\delta E-\delta V}{E-V}\right)} \approx p_0 + \frac{m}{p_0}(\delta E-\delta V)\,. \tag{3}$$

Substituting this into eq. (2) gives

$$\oint \frac{m}{p_0}(\delta E_n - \delta V)\,dx = 0\,, \tag{4}$$

or

$$\delta E_n \oint \frac{dx}{p_0} = \oint \delta V \frac{dx}{p_0}\,. \tag{5}$$

2. The period of oscillations is given by

$$T_n = \oint dt = \oint \frac{dx}{v_n} = \oint \frac{\partial p_n}{\partial E_n} dx\,, \tag{6}$$

where x, v_n, and E_n are classical coordinate, velocity and energy of the particle on the n'th energy level correspondingly. Therefore,

$$\frac{\delta E_n}{m}\oint dt = \frac{1}{m}\oint \delta V dt \quad \Rightarrow \delta E_n = \frac{1}{T_n}\int_0^{T_n} \delta V[x_n(t)]dt\,. \tag{7}$$

2. Rosen-Morse potential

1. We have to compute the integral

$$J = \int_{-x^*}^{x^*} \sqrt{2m(E - V(x)}\, dx\ , \quad V(x) = -\frac{V_0}{\cosh^2\left(\frac{x}{x_0}\right)}\ , \tag{8}$$

where $\pm x^*$ are the turning points and $-V_0 < E < 0$. The integral J can be transformed to a simple form by taking the derivative with respect to E. To the LO, one can neglect the dependence of the turning points on energy and set $\partial x^*/\partial E = 0$. Then,

$$\begin{aligned} \frac{dJ}{dE} &= m \int_{-x^*}^{x^*} \frac{dx}{\sqrt{2m(E - V(x)}} \\ &= \frac{mx_0}{\sqrt{-2mE}} \int_{-z^*}^{z^*} \frac{dz}{\sqrt{-(1+z^2)\left(1 + \frac{V_0}{E}\frac{1}{1+z^2}\right)}} \\ &= \frac{mx_0}{\sqrt{-2mE}} \int_{-z^*}^{z^*} \frac{dz}{\sqrt{\frac{V_0}{E} + 1}\sqrt{-1 - \frac{z^2}{\frac{V_0}{E} + 1}}} \\ &= \frac{mx_0}{\sqrt{-2mE}} \left[\arcsin \frac{z}{\sqrt{-1 - \frac{V_0}{E}}}\right]_{z=-z^*}^{z=z^*} = \frac{mx_0\pi}{\sqrt{-2mE}}\ , \end{aligned} \tag{9}$$

where $z^* = \sinh \frac{x^*}{x_0}$, and in going to the second line we used the change of variables

$$z = \sinh \frac{x}{x_0}\ . \tag{10}$$

Integrating the answer, we obtain

$$J = -\pi x_0 \sqrt{-2mE} + \text{Const}\ . \tag{11}$$

Observing that $J(E = -V_0) = 0$, we find the constant to be Const $= \pi x_0 \sqrt{2mV_0}$. Finally,

$$J = \pi x_0 \sqrt{2m}\left(\sqrt{V_0} - \sqrt{-E}\right) = \pi\hbar\left(n + \frac{1}{2}\right)\ , \tag{12}$$

and the energy levels are given by

$$E_n = -\left(\sqrt{V_0} - \frac{\hbar}{x_0\sqrt{2m}}\left(n + \frac{1}{2}\right)\right)^2, \quad n = 0, 1, 2, ... \tag{13}$$

Setting

$$x_0 = 2m = \hbar = 1\ , \quad V_0 = \frac{49}{4}\ , \tag{14}$$

we get

$$E_n = -(3 - n)^2 \quad\Rightarrow\quad E_0 = -9\ , \quad E_1 = -4\ , \quad E_2 = -1\ . \tag{15}$$

According to the exact answer, the three eigenvalues written above exhaust the discrete energy spectrum of the Rosen-Morse potential.

2. From eq. (15) we see that at $n = 3$ the energy of the bound state hits zero. This result cannot be trusted, since the LO WKB fails to reproduce the levels whose energy equals the asymptotics of the potential of the exponential type (see Problem 3 of Problem Set 5). In fact, there cannot be any bound states with the energies coinciding with the asymptotics of the potential at $x \to \pm\infty$ (provided that these asymptotics are the same). Hence, the $n = 3$-level is fake. All $n > 3$-levels are clearly fake too (they are either degenerate with the levels at $n = 0, 1, 2$, which is impossible for the bound states, or have the energy smaller than the minimum of the potential).

3. Tunneling through a parabolic barrier

1. The transmission coefficient D is given by

$$D = e^{\frac{i\pi}{2}} e^{-\frac{1}{\hbar}\int_{-x^*}^{x^*} p dx} , \tag{16}$$

where $\pm x^*$ are the turning points. Computation of the tunneling exponent is straightforward :

$$\begin{aligned}
\int_{-x^*}^{x^*} p dx &= \int_{-x^*}^{x^*} \sqrt{2m\left(V_0\left(1 - \frac{x^2}{x_0^2}\right) - E\right)} dx \\
&= x_0\sqrt{2mV_0}\int_a^b \sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}} \frac{dx}{x_0} \\
&= x_0\sqrt{2mV_0}\frac{1}{2}\left[\frac{x}{x_0}\sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}} + \left(1 - \frac{E}{V_0}\right)\text{atan}\frac{x/x_0}{\sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}}}\right]_{x=-x^*}^{x=x^*} \\
&= x_0\sqrt{2mV_0}\frac{\pi}{2}\left(1 - \frac{E}{V_0}\right) ,
\end{aligned} \tag{17}$$

where we used the relation $1 - (\pm x^*)^2/x_0^2 = E/V_0$. Finally,

$$D(E) = e^{\frac{i\pi}{2}} e^{-\frac{x_0}{2\hbar}(1-E/V_0)\pi\sqrt{2mV_0}} . \tag{18}$$

2. The WKB approach reproduces well the value of the transmission coefficient provided that the tunneling exponent is large. Denoting $\delta E = V_0 - E$, we have from eq. (18),

$$\delta E \gg \frac{\hbar}{x_0}\sqrt{\frac{2V_0}{m}} . \tag{19}$$

4. Lifetime in a cubic potential

The lifetime of a particle in the well is determined by the tunneling probability, $|D(E)|^2$. If the energy E of the particle is small compared to the height of the barrier,

$$E \ll V_{max} = \frac{4}{27} V_0 x_0^3 \,, \tag{20}$$

then the integral in $D(E)$ can be easily computed,

$$\begin{aligned}
\int_a^b p dx &= \int_a^b \sqrt{2m\left(V_0(x^2 x_0 - x^3) - E\right)}\, dx \\
&= \sqrt{2mV_0} \int_a^b \sqrt{x^2 x_0 - x^3 - \frac{E}{V_0}}\, dx \\
&\approx \sqrt{2mV_0}\frac{2}{15}\left[\sqrt{x_0 - x}(3x^2 - x_0 x - 2x_0^2)\right]_0^{x_0} \\
&= \frac{4}{15}\sqrt{2mV_0} x_0^{5/2} \,.
\end{aligned} \tag{21}$$

In going to the third line, we neglected the term E/V_0, since its contribution to the integral becomes essential only near the turning points a and b, where the integrand is small. For the same reason, we approximated the limits of integration as $a \approx 0$, $b \approx x_0$.

The lifetime is given by $\tau = T/|D(E)|^2$, where T is the period of classical oscillations in the well,

$$\begin{aligned}
T = \oint dt = 2\int_{a_1}^{a_2} \frac{m dx}{p(x)} &\approx \sqrt{\frac{2m}{E}} \int_{a_1}^{a_2} \frac{dx}{\sqrt{1 - \frac{x_0 V_0}{E} x^2}} \\
&= \sqrt{\frac{2m}{x_0 V_0}} \left[\arcsin \sqrt{\frac{x_0 V_0}{E}} x\right]_{a_1}^{a_2} = \sqrt{\frac{2m}{x_0 V_0}} \pi \,.
\end{aligned} \tag{22}$$

Again, since $E \ll V_0$, we used the quadratic approximation for the potential near the bottom of the well,

$$V \approx V_0 x_0 x^2 \,, \quad x \ll x_0 \,, \tag{23}$$

and the turning points a_1, a_2 of the classical oscillations are determined from the equation

$$E = V_0 x_0 a_{1,2}^2 \,, \tag{24}$$

which gives $a_1 = -a_2 = \sqrt{\frac{E}{V_0 x_0}}$. From eqs. (21) and (22) we have

$$\tau = \sqrt{\frac{2m}{x_0 V_0}} \pi \exp\left(\frac{8}{15\hbar}\sqrt{2mV_0} x_0^{5/2}\right) \,. \tag{25}$$

5*. Super-WKB approach

1. We substitute the relation $V = W^2 - \frac{\hbar}{\sqrt{2m}}W'$ into the Bohr-Sommerfeld quantization condition and expand to the LO in $\hbar$. This gives,

$$\int_a^b \sqrt{2m(E-U)}\,dx = \int_a^b \sqrt{2m(E-W^2)}\,dx + \frac{\hbar}{2}\int_a^b \frac{W'}{\sqrt{E-W^2}}dx + O(\hbar^2) \\ = \pi\hbar\left(n+\frac{1}{2}\right) . \tag{26}$$

By a and b we now understand the turning points of the function W^2. Assuming

$$W(a) = -W(b) , \tag{27}$$

one finds

$$\frac{\hbar}{2}\int_a^b \frac{W'}{\sqrt{E-W^2}}dx = \frac{\hbar}{2}\arcsin\frac{W}{\sqrt{E}}\bigg|_a^b = \frac{\pi\hbar}{2} + O(\hbar^2) . \tag{28}$$

Thus, to the LO

$$\int_a^b \sqrt{2m(E-W^2)}\,dx = \pi\hbar n , \quad n = 0, 1, 2, ... \tag{29}$$

This is the so-called LO Super-WKB (or SWKB for short) quantization condition. If, on the other hand, $W(a) = W(b)$, the term (28) vanishes, and we have

$$\int_a^b \sqrt{2m(E-W^2)}\,dx = \pi\hbar\left(n+\frac{1}{2}\right) . \tag{30}$$

2. Here we assume that the superpotential obeys eq. (27). Since the integrand in eq. (28) is nonnegative, at $n = 0$ one must have $a = b$. But then $W(a) = W(b) = 0$, and

$$E_0 = W(a)^2 = W(b)^2 = 0 . \tag{31}$$

3. The quantum systems whose superpotential possesses the property (27) are called supersymmetric. It turns out that for such systems, the energy of the ground state is exactly zero. Hence, according to eq. (31), the LO SWKB is exact at $n = 0$. This makes it differ from the standard WKB whose accuracy is supposed to be low at low n. Therefore, one can expect the SWKB to give better predictions for supersymmetric potentials. To check this, consider the following potential,

$$V(x) = -\frac{1}{x} + \frac{x(x+2)}{(1+x+\frac{1}{2}x^2)^2} + \frac{1}{16} . \tag{32}$$

The corresponding superpotential is given by (see figure 1) [1]

$$W(x) = \frac{x^6 - 16x^4 - 56x^3 - 108x^2 - 240x - 192}{4x(x^2+2x+2)(x^3+6x^2+18x+24)} . \tag{33}$$

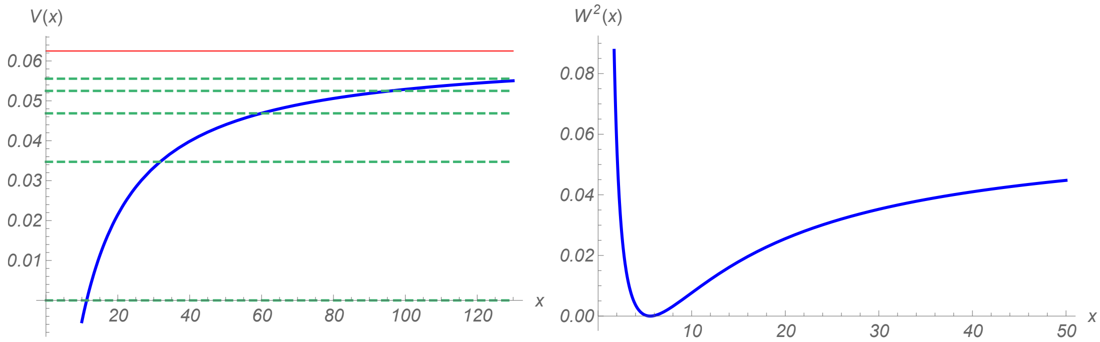

FIGURE 1 – The left panel : the potential (32). Shown also are the asymptotics of the potential at infinity (the red line), and the first five eigenvalues. The right panel : the superpotential (33).

n	$E_n \times 10^2$	WKB	SWKB
0	0	−47.002	0
1	3.472	1.644	3.460
2	4.688	4.019	4.682
3	5.25	4.931	5.247
4	5.556	5.407	5.554
5	5.740	5.632	5.739
6	5.859	5.788	5.859
7	5.941	5.892	5.941
8	6	5.965	6.000
9	6.043	6.017	6.043

TABLE 1 – The first 10 eigenvalues of the potential (32).

Here, for simplicity, we move to dimensionless notations and set $2m = \hbar = 1$. Now we compute numerically the energy levels by the means of the Bohr-Sommerfeld quantization condition, by the formula (29), and by the exact formula

$$E_n = \frac{1}{16}\frac{n(n+4)}{(n+2)^2}\,, \qquad n = 0, 1, 2, ... \tag{34}$$

The results are summarized in table 1. We see that the LO SWKB achieves the accuracy 10^{-3} at $n = 5$, at which the standard WKB can only give the accuracy $\approx 2 \cdot 10^{-2}$. Note, however, that for the potentials without supersymmetry (in particular, for which $E_0 \neq 0$), the convergence of the SWKB method is not much superior to that of the WKB.

6*. Scattering off the peak

As the energy E of the particle approaches zero, the turning points, where its wave function crosses the potential barrier, converge to the single point $x_0 = 0$. Hence, for small

1. For details, see DeLaney, David *et al*. Phys.Lett. B247 (1990) 301-308.

enough energies it makes sense to continue the wave function of the particle directly from one classically allowed region to another, thus avoiding the classically forbidden region. For this to be possible, one has to construct a region in the plane of complex x, where the wave function can be approximated by the WKB expressions of the plane-wave type.

1. Near the origin, the potential V can be expanded as

$$V(x) = \frac{kx^2}{2} + bx^3 + O(x^4) . \tag{35}$$

It will be convenient to think of $k \equiv V''(0)$ as the curvature of the potential at zero point. Then, $b \sim k'(0) \equiv k'$. To neglect the third-order term in eq. (35), one has to make sure that

$$|bx^3| \ll |kx^2| \quad \Rightarrow \quad |x| \ll \left|\frac{k}{k'}\right| . \tag{36}$$

In other words, $|x|$ must be small compared to the characteristic length of change of the curvature.

Next, we expand the momentum in the ratio $|E/V(x)|$,

$$p(x) = \sqrt{2m(E - V(x))} \approx \sqrt{-2mV(x)}\left(1 - \frac{E}{2V(x)}\right) . \tag{37}$$

For this expansion to work, one should require $|E/V(x)| \ll 1$. Applying the quadratic approximation for the potential, we rewrite this condition as

$$|x| \gg \left|\frac{E}{k}\right|^{\frac{1}{2}} . \tag{38}$$

Overall, the region $\mathcal{R}$ is defined by the conditions (36) and (38) :

$$\left|\frac{E}{k}\right|^{\frac{1}{2}} \ll |x| \ll \left|\frac{k}{k'}\right| . \tag{39}$$

Clearly, this inequality can be satisfied for some $|x|$ provided that $|E|$ is small enough.

2. First, we should check that the LO WKB approximation is applicable in the region $\mathcal{R}$. Making use of eqs. (37) and (35), we have

$$|\lambda\!\!\!^{-}{}'| \ll 1 \quad \Rightarrow \quad |x| \gg \left|\frac{\hbar^2}{k}\right| . \tag{40}$$

This is compatible with the inequality (39) if

$$\left|\frac{d}{dx}\frac{1}{k}\right| \ll \hbar^{-2} , \tag{41}$$

that is, the characteristic length at which the curvature radius changes must be large in Planck units. With this assumption, we can use the LO WKB in the region $\mathcal{R}$.

To the right side from the turning point, we have the wave function transmitted through the barrier,

$$\psi_{x>0} = \frac{d}{\sqrt{p}} e^{\frac{i}{\hbar}\int p dx} \approx d(mk)^{-\frac{1}{4}} x^{\frac{i}{\hbar}E\sqrt{\frac{m}{k}}-\frac{1}{2}} e^{\frac{i}{\hbar}\frac{x^2}{2}\sqrt{mk}} . \tag{42}$$

Note the unusual dispersion relation for this function : x enters quadratically in the phase factor. If we restore the time-dependence of the wave function, the phase factor becomes

$$e^{\frac{i}{\hbar}\left(\frac{x^2}{2}\sqrt{mk}-Et\right)} . \tag{43}$$

To keep the phase profile unchanged, x^2 must increase as the time increases. For $x > 0$, this implies the growth of x. Hence, eq. (42) describes the right-moving wave, as it should.

Let us now write the wave function to the left side from the turning point,

$$\begin{aligned}\psi_{x<0} &= \frac{r}{\sqrt{p}} e^{\frac{i}{\hbar}\int p dx} + \frac{q}{\sqrt{p}} e^{-\frac{i}{\hbar}\int p dx} \\ &\approx r(mk)^{-\frac{1}{4}}(-x)^{\frac{i}{\hbar}E\sqrt{\frac{m}{k}}-\frac{1}{2}} e^{\frac{i}{\hbar}\frac{x^2}{2}\sqrt{mk}} + q(mk)^{-\frac{1}{4}}(-x)^{-\frac{i}{\hbar}E\sqrt{\frac{m}{k}}-\frac{1}{2}} e^{-\frac{i}{\hbar}\frac{x^2}{2}\sqrt{mk}} .\end{aligned} \tag{44}$$

The first term in this expression is characterized by the phase factor (43). But now $x < 0$, so, to keep the phase unchanged, x must run towards $-\infty$ as the time increases. Hence, the first term represents the left-moving wave, that is, the wave reflected from the barrier. Similarly, we conclude that the second term describes the incident wave, hence we set $q = 1$.

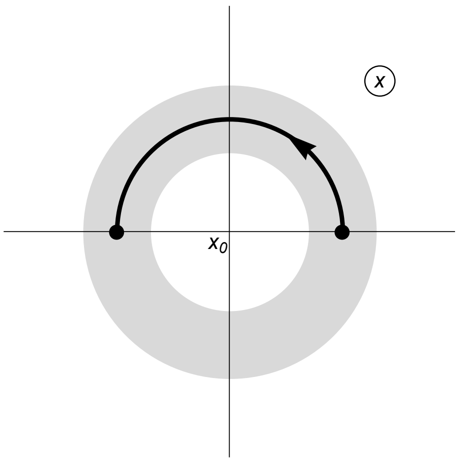

Figure 2 – The region $\mathcal{R}$ in the plane of complex x, and the continuation contour.

3. Continued to the complex plane, $\mathcal{R}$ becomes a ring centered around the turning point (see figure 2). We take the function (42) and continue it to the region $x < 0$ along the path in $\mathcal{R}$ going in the upper semiplane. This amounts to the replacement

$$x|_{x>0} = Re^{i\cdot 0} \to Re^{i\pi} = (-x)e^{i\pi}|_{x<0} . \tag{45}$$

Making this replacement, we restore the first term of the function (44) and obtain the connection formula

$$r = de^{-\frac{\pi}{\hbar}E\sqrt{\frac{m}{k}}} e^{-\frac{i\pi}{2}} . \tag{46}$$

It gives one relation between the reflection and transmission coefficients $R = |r|^2$, $D = |d|^2$. Combining it with the unitarity condition, we have

$$\begin{cases} R = De^{-2\epsilon} \\ R + D = 1 \end{cases} , \quad \epsilon = \frac{\pi}{\hbar} E \sqrt{\frac{m}{k}} . \tag{47}$$

The solution of this system is

$$D = \frac{1}{1 + e^{-\epsilon}} , \quad R = \frac{e^{-\epsilon}}{1 + e^{-\epsilon}} . \tag{48}$$

Recall that these formulas are valid while the inequalities (39) and (41) fulfill.

4. Taking the limit $\epsilon \to 0$ in eq. (48), we find

$$R = D = \frac{1}{2} . \tag{49}$$

This nice result tells us that the quantum particle scattering off the parabolic peak has equal probabilities to slide forward and to bounce back.