QUANTUM PHYSICS III

Solutions to Problem Set 1 9 September 2025

1. Gaussian Integrals

To compute the first integral, one can use the following trick. First, multiply I_1 by itself,

$$I_1^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\, e^{-\frac{x^2+y^2}{2}} \ , \tag{1}$$

and then compute the obtained expression using the polar coordinates,

$$I_1^2 = 2\pi \int_0^{\infty} dr\, re^{-\frac{r^2}{2}} = 2\pi \ . \tag{2}$$

Hence,

$$I_1 = \sqrt{2\pi} \ . \tag{3}$$

To compute I_2 and I_3, we first note that

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}} \ , \tag{4}$$

and

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x-x_0)^2} = \int_{-\infty}^{\infty} dx'\, e^{-\frac{1}{2}x'^2} = I_1 \ . \tag{5}$$

So, we have

$$I_2 = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+bx} = \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}\left(x-\frac{b}{a}\right)^2+\frac{b^2}{2a}} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}} \ , \tag{6}$$

$$I_3 = \int_{-\infty}^{\infty} x^2 e^{-\frac{ax^2}{2}} = -2\frac{d}{da}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} = -2\frac{d}{da}\sqrt{\frac{2\pi}{a}} = \frac{1}{a}\sqrt{\frac{2\pi}{a}} \ . \tag{7}$$

2. A Gaussian packet

1. Since the wave function $\Psi(p,0) = \frac{A}{(2\pi)^{1/4}} e^{-\frac{\sigma^2}{\hbar^2}(p-p_0)^2}$ must be normalized to one, it follows that

$$\int_{-\infty}^{\infty} |\Psi(p,0)|^2 dp = 1 \ \Rightarrow\ A = \sqrt{\frac{2\sigma}{\hbar}} \ . \tag{8}$$

2. Recall that by definition the Fourier image is (note our convention about 2π multipliers)

$$\Psi(x,0) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dp\, \Psi(p,0) e^{\frac{i}{\hbar}px} \,. \tag{9}$$

Substituting the expression for $\Psi(p,0)$, we obtain,

$$\Psi(x,0) = \frac{\sqrt{2\sigma/\hbar}}{(2\pi)^{3/4}} \int_{-\infty}^{\infty} dp\, e^{-\frac{\sigma^2 p^2}{\hbar^2} + \left(\frac{2\sigma^2 p_0}{\hbar} + ix\right)\frac{p}{\hbar}} e^{-\frac{\sigma^2 p_0^2}{\hbar^2}} = \frac{1}{(2\pi)^{1/4}\sigma} e^{\frac{i}{\hbar}p_0 x} e^{-\frac{x^2}{4\sigma^2}} \,. \tag{10}$$

Thus, $\Psi(x,0)$ is again a Gaussian function.
The dispersion of the operator A is defined as

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \,. \tag{11}$$

Let us first compute $\Delta p(0)$. We have

$$\langle p(0) \rangle = \int_{-\infty}^{\infty} dp\, p|\Psi(p,0)|^2 = \int_{-\infty}^{\infty} dp\, (p+p_0)|\Psi(p+p_0,0)|^2 = p_0 \,, \tag{12}$$

because the expression $p\,|\Psi(p+p_0,0)|^2$ is an odd function of p. Hence,

$$(\Delta p(0))^2 = \int_{-\infty}^{\infty} dp\, p^2|\Psi(p,0)|^2 - p_0^2 = \int_{-\infty}^{\infty} dp\, p^2|\Psi(p+p_0,0)|^2 + p_0^2 - p_0^2 = \hbar^2/4\sigma^2 \,, \tag{13}$$

and

$$\Delta p(0) = \hbar/2\sigma \,. \tag{14}$$

Similarly, using the expression (10) for $\Psi(x,0)$, one obtains

$$\Delta x(0) = \sigma \,. \tag{15}$$

Of course, the above expressions for the dispersions can be seen immediately from the expressions for $\Psi(p,0)$ and $\Psi(x,0)$, since the latter are Gaussian functions. Note that $\Delta x(0)\Delta p(0) = \hbar/2$, that is, the state $|\Psi\rangle$ minimizes the uncertainty relation for the pair of operators $x(0)$, $p(0)$.

3. Solving the Cauchy problem gives

$$\Psi(p,t) = \Psi(p,0) e^{-\frac{i}{\hbar}\omega(p)t} \,, \quad \omega(p) = \frac{p^2}{2m} \,. \tag{16}$$

To find $\Psi(x,t)$, we make the Fourier transform of $\Psi(p,t)$ and use the results of exercise 1. We have

$$\Psi(x,t) = \frac{\sqrt{2\sigma}}{(2\pi)^{3/4}} \int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2}\left(2\sigma^2 + \frac{i\hbar t}{2m}\right)\frac{p^2}{\hbar^2} + \left(\frac{2\sigma^2 p_0}{\hbar} + ix\right)\frac{p}{\hbar}} e^{-\frac{\sigma^2 p_0^2}{\hbar^2}} \tag{17}$$

Doing the integral in p will result in an exponent which takes the form

$$\left(\frac{2\sigma^2 p_0}{\hbar^2} + \frac{ix}{\hbar}\right)^2 \Big/ \left(\frac{2\sigma^2}{\hbar^2} + \frac{it}{m\hbar}\right) - \frac{\sigma^2 p_0^2}{\hbar^2} \tag{18}$$

and of course some new normalisation constant in front. The reason that we focus on the exponent is due to that we know this in the end will result in a Gaussian, and the dispersion Δx can be read directly from the quadratic coefficient of x.
Carry on, we read the second order term in x in the exponent from eq.(18),

$$\frac{(2\sigma^2 p_0/\hbar + ix)^2}{2\sigma^2 + it\hbar/m} = \frac{-x^2 + 4i\sigma^2 p_0/\hbar + 4\sigma^4 p_0^2/\hbar^2}{2\sigma^2 + i\hbar t/m} \tag{19}$$

and the denominator of the real part is $4\sigma^4 + \hbar^2 t^2/m^2$, with which (and eq.(16)) we identify the dispersion

$$\Delta p(t) = \Delta p(0)\ , \quad \Delta x(t) = \Delta x(0)\sqrt{1 + \frac{\hbar^2 t^2}{4m^2\sigma^4}}\ . \tag{20}$$

When t increases, $\Delta x(t)$ grows as well, and the equality $\Delta x \Delta p = \hbar/2$ is not valid anymore : the wave packet is spreading. From eq. (17) we note also that p_0/m is nothing but the group velocity of the wave packet.

3. Quantum fluctuations

We assume that the system comprising the hill and the object on the top of it does not interact with anything. This allows us to drop out the potential term in the Schrödinger equation. The wave function of the object can now be taken Gaussian, like in the previous exercise. On physical grounds we expect that the probability density for the object to fall from the top of the hill is saturated at the time when its dispersion $\Delta x(t)$ becomes equal to the size of the top l. The dispersion is given by (see eq. (20))

$$(\Delta x(t))^2 = \sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2}\ . \tag{21}$$

1. Denote the fall time by t_0. Consider the equality $\Delta x(t_0) = l$ as an equation for $t_0 = t_0(\sigma)$:

$$\left(\frac{\hbar t_0}{2m}\right)^2 = -\sigma^4 + l^2\sigma^2. \tag{22}$$

   Maximizing t_0 with respect to σ gives

$$\sigma = \frac{l}{\sqrt{2}}\ , \quad t_{0\,max} = \frac{ml^2}{\hbar} \sim \frac{10^{-3}kg \cdot 10^{-4}m^2}{10^{-34}\frac{kg\cdot m^2}{s}} = 10^{27}s\ . \tag{23}$$

2. Substituting $\sigma = 10^{-9}cm \ll l$ into eq. (22) gives

$$t_0 \approx \frac{ml\sigma}{\hbar} = 10^{18}s\ . \tag{24}$$

4. Harmonic oscillator

1. Recall that $H = \hbar\omega(a^\dagger a + 1/2)$ and $[a, a^\dagger] = 1$. We have

$$[a, H] = \hbar\omega[a, a^\dagger a + 1/2] = \hbar\omega[a, a^\dagger a] = \hbar\omega\left(aa^\dagger a - a^\dagger aa\right) \\ = \hbar\omega[a, a^\dagger]a = \hbar\omega a \ . \tag{25}$$

   Similarly, $[a^\dagger, H] = -\hbar\omega a^\dagger$.
   In the Heisenberg picture,

$$i\hbar\frac{da}{dt} = [a, H] = \hbar\omega a \quad \Rightarrow \quad a(t) = a(0)e^{-i\omega t} \ . \tag{26}$$

   Therefore,

$$[a(t), a^\dagger(t)] = [a(0)e^{-i\omega t}, a^\dagger(0)e^{i\omega t}] = [a(0), a^\dagger(0)] = 1 \ . \tag{27}$$

2. One should express $x(t)$ and $p(t)$ through $\alpha(t)$ and $\alpha^*(t)$,

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}}\,(\alpha(t) + \alpha^*(t)) \ , \\ p(t) = -i\sqrt{\frac{\hbar m\omega}{2}}\,(\alpha(t) - \alpha^*(t)) \ , \tag{28}$$

   and put them into H,

$$H = \frac{1}{2m}p(x)^2 + \frac{1}{2}m\omega^2 x(t)^2 = \frac{1}{2m}p(0)^2 + \frac{1}{2}m\omega^2 x(0)^2 \\ = \hbar\omega\frac{1}{4}4\alpha(0)\alpha^*(0) = \hbar\omega|\alpha(0)|^2 \ , \tag{29}$$

   where we made use of the conservation of H with time.

5. Gaussian integrals in more dimensions

1. The idea is to diagonalize the matrix A in order to reduce the integral into the product of integrals of Gaussian functions. Let O be the desired orthogonal transformation,

$$O^{-1}AO = \mathrm{diag}(\lambda_1, ..., \lambda_N) \ , \tag{30}$$

   where λ_i are eigenvalues of A. The corresponding change of variables reads as follows,

$$y = O^t x \quad \Rightarrow \quad dy_1...dy_N = \det O \cdot dx_1...dx_N = dx_1...dx_N \ , \tag{31}$$

   since O is orthogonal. Applying the transformation (31) to the integral gives

$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1...dx_N \ \exp\left(-\frac{1}{2}x^t A x + B^t x\right) \\ = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dy_1...dy_N \ \exp\left(-\frac{1}{2}y^t\left(O^{-1}AO\right)y + B^t O y\right) \\ = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dy_1...dy_N \ \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\lambda_i y_i^2 + B^t O y\right) \\ = \sqrt{\frac{(2\pi)^N}{\Pi_{i=1}^N \lambda_i}}\exp\left(\frac{1}{2}\sum_{i=1}^{N} B_j O_{ji}\lambda_i^{-1}(O^{-1})_{ik}B_k\right) = \sqrt{\frac{(2\pi)^N}{\det A}}\exp\left(\frac{1}{2}B^t A^{-1} B\right) \ . \tag{32}$$

2. Here one can use the following trick. First, given the exponent $\exp\left(-\frac{1}{2}x^tAx\right)$, we supplement it with the "source" $\exp(B^tx)$ of the variable x. Then we differentiate the source with respect to B_i to obtain the factor x_i in the integrand. Finally, at the end of calculation we take the limit $B = 0$. Here is the implementation of this program :

$$
\begin{aligned}
&\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1...dx_N \; x_{i_1}x_{i_2} \exp\left(-\frac{1}{2}x^tAx\right) \\
=&\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1...dx_N \; x_{i_1}x_{i_2} \exp\left(-\frac{1}{2}x^tAx + B^tx\right)\Bigg|_{B=0} \\
=&\frac{d}{dB_{i_1}}\frac{d}{dB_{i_2}} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} dx_1...dx_N \; \exp\left(-\frac{1}{2}x^tAx + B^tx\right)\Bigg|_{B=0} \\
=&\sqrt{\frac{(2\pi)^N}{\det A}}\frac{d}{dB_{i_1}}\frac{d}{dB_{i_2}} \exp\left(\frac{1}{2}B_i\left(A^{-1}\right)_{ij}B_j\right)\Bigg|_{B=0} = \sqrt{\frac{(2\pi)^N}{\det A}}\left(A^{-1}\right)_{i_1i_2} \; .
\end{aligned}
\tag{33}
$$

Thus,

$$
\langle x_{i_1}x_{i_2}\rangle = \left(A^{-1}\right)_{i_1i_2} \; . \tag{34}
$$

As for the average $\langle x_{i_1}x_{i_2}x_{i_3}x_{i_4}\rangle$, it is obtained by simply adding more differentials d/dB_{i_p} with various i_p, and the result

$$
\langle x_{i_1}x_{i_2}x_{i_3}x_{i_4}\rangle = \langle x_{i_1}x_{i_2}\rangle\langle x_{i_3}x_{i_4}\rangle + \langle x_{i_1}x_{i_3}\rangle\langle x_{i_2}x_{i_4}\rangle + \langle x_{i_1}x_{i_4}\rangle\langle x_{i_2}x_{i_3}\rangle \tag{35}
$$

is reproduced straightforwardly. Note finally that any odd number of derivatives inevitably gives some B_{i_p} appearing before the exponent, hence, after setting $B = 0$, all such terms vanish, and

$$
\langle x_{i_1}x_{i_2}...x_{i_k}\rangle = 0 \; , \quad \text{if } k \text{ is odd} \; . \tag{36}
$$

6. A pen

At first glance, this problem seems quite similar to the Exercise 3. However there is an important difference : in Exercise 3, the top of the hill was flat, i.e. the potential energy surface was flat, at least in some neighborhood in phase space. It is not the case here : even though the pen standing on its tip is a classical equilibrium, any small perturbation will change the potential energy and make the pen fall.

It is instructive to first solve this problem classically. Consider a pen of length L and mass m (distributed uniformly along the length). The only degree of freedom here is the tilt angle θ, measurde relative to the vertical axis. One can use e.g. the Lagrangian formalism to get an equation of motion :

$$
\mathcal{T} = \frac{1}{2}\int_{l=0}^{l=L} \mathrm{d}m l^2\dot{\theta}^2 \tag{37}
$$

$$
= \frac{m}{2L}\int_{l=0}^{l=L} \mathrm{d}l \; l^2\dot{\theta}^2 \tag{38}
$$

$$
= \frac{mL}{6}\dot{\theta}^2 \tag{39}
$$

$$\mathcal{V} = g \int_{l=0}^{l=L} \mathrm{d}m \; l \cos\theta \tag{40}$$

$$= \frac{mg}{L} \int_{l=0}^{l=L} \mathrm{d}l \; l \cos\theta \tag{41}$$

$$= \frac{mgL\cos\theta}{2} \tag{42}$$

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \tag{43}$$

$$= \frac{1}{6} mL^2\dot{\theta}^2 - \frac{mgL\cos\theta}{2} \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \tag{45}$$

$$\Leftrightarrow \frac{1}{3} mL^2\ddot{\theta} = \frac{mgL}{2}\sin\theta \tag{46}$$

$$\Leftrightarrow \ddot{\theta} = \frac{3g}{2L}\sin\theta \tag{47}$$

$$\Leftrightarrow \ddot{\theta} = \omega^2 \sin\theta \tag{48}$$

where $\omega^2 \equiv 3g/2L$. We have an equation of motion similar to that of a pendulum, except the sign is reversed. This is what gives the highly diverging behavior of our pencil, instead of the oscillatory one of a normal pendulum.

To compute the time it would take pencil to fall, one would need to solve this differential equation, and find the time T at which $\theta(T) = \pi/2$. This is impossible to do analytically. However this is a standard second order homogeneous differential equation, so its solutions are all of the form :

$$\theta(t) = \theta_0 f_1(\omega t) + \frac{\omega_0}{\omega} f_2(\omega t) \tag{49}$$

with θ_0 and ω_0 the initial tilt angle and angular velocity respectively.

But this is all classical, where does the quantumness comes in ? A pen is big enough to be accurately described with the classical equation of motion ; however, there is one moment when quantum mechanics will matter : the initial conditions. One would be tempted to start with $\theta_0 = \omega_0 = 0$, i.e. zero tilt angle and zero angular velocity. This would lead to an infinite time to fall ; however quantum fluctuations render this case impossible.

The tilt angle θ and the angular momentum $L = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mL^2\dot{\theta}/3$ are canonically conjugate by defininition, so they obey the Heisenberg uncertainty relation :

$$\Delta\theta\Delta L \geq \frac{\hbar}{2} \tag{50}$$

This means that at best, the initial conditions satisfy the following :

$$\theta_0\omega_0 = \frac{3\hbar}{2mL^2} \equiv A^2 \tag{51}$$

We can parameterize these solutions with $\alpha \in]0, +\infty[$:

$$\theta_0 = A\alpha \tag{52}$$

$$\omega_0 = \frac{A}{\alpha} \tag{53}$$

These can be put back into the equations of motion to yield

$$\theta_\alpha(t) = A\left(\alpha f_1(\omega t) + \frac{1}{\alpha\omega} f_2(\omega t)\right) \tag{54}$$

Unfortunately, the rest cannot be done analytically. We have to solve $\theta_\alpha(T) = \theta_{\text{max}} = \pi/2$ for different values of α, to find the α that maximizes T. Solving the differential equation can be done numerically, which for $m = 5$ g and $L = 10$ cm yields a maximal time of 3.5 seconds.

Remark : details of how to do this numerically is beyond the scope of this course. For whose who are curious, this blogpost goes more into the details. (it is also from this reference that the value of 3.5 seconds was taken).