
QUANTUM PHYSICS III

Solutions to Problem Set 11

25 November 2025

1. Properties of spherical Bessel functions

1. We can use the Taylor series for $\sin \rho = \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n+1}}{(2n+1)!}$, and the fact that :

$$\left(\frac{1}{\rho} \frac{d}{d\rho}\right)^l \rho^{2n} = \begin{cases} \frac{(2n)!!}{(2n-2l)!!} \rho^{2n-2l} & \text{for } n \geq l, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

which can easily be proved by induction. We thus have a series expansion for j_l :

$$j_l(\rho) = \sum_{n=l}^{\infty} \frac{(-1)^{n+l} (2n)!!}{(2n+1)!(2n-2l)!!} \rho^{2n-l}. \quad (2)$$

As $\rho \rightarrow 0$, we can keep only the leading order ($n = l$), hence :

$$j_l(\rho) \sim \frac{(2l)!! \rho^l}{(2n+1)!} = \frac{\rho^l}{(2l+1)!!}. \quad (3)$$

The case of y_l is similar, expanding the cosine in a Taylor series.

2. We can prove this by induction. The relation holds for $l = 0$:

$$j_0(\rho) = \frac{\sin \rho}{\rho}, \quad (4)$$

$$y_0(\rho) = -\frac{\cos \rho}{\rho}. \quad (5)$$

Assuming the relation holds for a given l , we have :

$$\begin{aligned} j_{l+1}(\rho) &= (-1)^{l+1} \rho^{l+1} \left(\frac{1}{\rho} \frac{d}{d\rho}\right)^{l+1} \frac{\sin \rho}{\rho} \\ &= (-1)^{l+1} \rho^{l+1} \left(\frac{1}{\rho} \frac{d}{d\rho}\right) (-1)^l \rho^{-l} j_l(\rho) \\ &\sim -\rho^l \frac{d}{d\rho} \rho^{-l-1} \sin(\rho - l\frac{\pi}{2}) \\ &= -(-l-1) \rho^{-2} \sin(\rho - l\frac{\pi}{2}) - \rho^{-1} \cos(\rho - l\frac{\pi}{2}) \\ &\sim \frac{1}{\rho} \sin(\rho - (l+1)\frac{\pi}{2}). \end{aligned}$$

In the last line, we neglected the first term, because $1/\rho^2$ decays faster than $1/\rho$ at large ρ . The asymptotic for y_l can be proved in a similar manner.

2. Scattering phase shift in a Yukawa potential

1. One has to start with the decomposition of the scattering amplitude in spherical harmonics :

$$f(\mathbf{p}' \leftarrow \mathbf{p}) = \sum_{l=0}^{\infty} (2l+1) f_l(p) P_l(\cos \theta). \quad (6)$$

With the orthogonality relation of the hint, it is trivial to see that

$$\begin{aligned} f_l &= \frac{1}{2} \int_0^{2\pi} d\theta \sin \theta f(\mathbf{p}' \leftarrow \mathbf{p}) P_l(\cos \theta) \\ &= -\frac{mV_0}{\mu} \int_0^{2\pi} d\theta \frac{P_l(\cos \theta) \sin \theta}{2p^2(1 - \cos \theta) + \mu^2} \\ &= -\frac{mV_0}{\mu} \int_{-1}^1 d\zeta \frac{P_l(\zeta)}{2p^2(1 - \zeta) + \mu^2} \\ &= -\frac{mV_0}{\mu} Q_l \left(1 + \frac{\mu^2}{2p^2} \right). \end{aligned}$$

For $|\delta_l| \ll 1$, we can expand f_l in first order in δ_l :

$$f_l = \frac{e^{i\delta_l} \sin(\delta_l)}{p} \approx \frac{\delta_l}{p}. \quad (7)$$

Thus :

$$\delta_l = -\frac{mV_0}{\mu p} Q_l \left(1 + \frac{\mu^2}{2p^2} \right). \quad (8)$$

2. (a) For $\zeta > 0$, $Q_l(\zeta)$ is positive. Thus when the potential is attractive, $V_0 < 0$, hence $\delta_l > 0$.
 (b) This condition translates to $1/p \gg 1/\mu \Leftrightarrow \mu \gg p$, thus $1 + \mu^2/2p^2 \approx \mu^2/2p^2$. In the series expansion, we can keep only the first term, since the next ones will be at least $1/\zeta^2$ smaller. We obtain :

$$\delta_l = -\frac{mV_0}{\mu p} \cdot \frac{2^{l+1} l! p^{2l+2}}{(2l+1)!! \mu^{2l+2}} = -\frac{2^{l+1} l! mV_0}{(2l+1)!! \mu^{2l+3}} \cdot p^{2l+1}. \quad (9)$$

3. Scattering off a spherical potential

1. Let us start with an impenetrable sphere

$$V(r) = \begin{cases} \infty & \text{for } r < R, \\ 0 & \text{for } r > R. \end{cases} \quad (10)$$

We will use the properties that the wave function must vanish at $r = R$ because the sphere is impenetrable. Therefore we have the following condition on the radial part of the wavefunction :

$$R_l(r)|_{r=R} = 0. \quad (11)$$

For $r > R$, the potential is zero. Hence, we can express R_l as a linear combination of spherical Bessel and von Neumann functions :

$$R_l(r) = \cos \delta_l j_l(kr) - \sin \delta_l y_l(kr). \quad (12)$$

Since we only care about the relative phase shift, we arbitrarily set the amplitude to 1. The boundary condition implies :

$$j_l(kR) \cos \delta_l - y_l(kR) \sin \delta_l = 0. \quad (13)$$

Let us consider the $l = 0$ case (s -wave scattering) specifically. The equation becomes

$$\tan \delta_0 = \frac{\sin kR/kR}{-\cos kR/kR} = -\tan kR. \quad (14)$$

The radial-wave function $R_0(r)$ varies as :

$$R_{l=0}(r) = \frac{\cos \delta_0 \sin kr - \sin \delta_0 \cos kr}{kr} = \frac{\sin(kr + \delta_0)}{kr}, \quad (15)$$

with $\delta_0 = -kR$. Therefore, if we plot $rR_{l=0}(r)$ as a function of distance r , we obtain a sinusoidal wave that is shifted when compared to the free sinusoidal wave by an amount R .

Let us now study the low- and high-energy limits. For $kR \ll 1$:

$$\begin{aligned} j_l(kr) &\approx \frac{(kr)^l}{(2l+1)!!}, \\ y_l(kr) &\approx -\frac{(2l-1)!!}{(kr)^{l+1}}. \end{aligned} \quad (16)$$

to obtain

$$\tan \delta_l = \frac{-(kR)^{2l+1}}{\{(2l+1)[(2l-1)!!]^2\}}. \quad (17)$$

It is therefore justified to ignore δ_l with $l \neq 0$. In other words, we have s -wave scattering only which is actually expected for almost any finite-range potential at low energy.

2. Because $\delta_0 = -kR$ regardless of whether kR is large or small, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2 \delta_0}{k^2} = R^2 \quad \text{for } kR \ll 1. \quad (18)$$

It is interesting that the total cross section given by

$$\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi R^2. \quad (19)$$

is four times the geometric cross section πR^2 . By geometric cross section we mean the area of the disc of radius R that blocks the propagation of the plane wave. Low-energy scattering, of course, means a very large-wavelength scattering, and we do not necessarily expect a classically reasonable result.

4. Scattering off a constant potential

1. Similar to the problem of a square potential in 1D, the idea is to solve the Schrödinger equation on both regions $r < R$ and $r > R$, and connect them by continuity.

For $r < R$, we have :

$$R_l(r) = A j_l(\kappa r), \quad (20)$$

with $0 < E - V_0 \equiv \frac{\hbar^2 \kappa^2}{2m}$. There is no contribution from the von Neumann function, because the wave function must not diverge at $r \rightarrow 0$.

For $r > R$, we have :

$$R_l(r) = B [\cos(\delta_l) j_l(kr) - \sin(\delta_l) y_l(kr)] \quad (21)$$

In order to find the three unknowns A , B and the scattering phase shift δ_l , we need to match both the wave function and its spatial derivative at the boundary between the two regions, $r = R$:

$$\begin{cases} A j_l(\kappa R) = B [\cos(\delta_l) j_l(kR) - \sin(\delta_l) y_l(kR)] \\ A \kappa j_l'(\kappa R) = B k [\cos(\delta_l) j_l'(kR) - \sin(\delta_l) y_l'(kR)] \end{cases} \quad (22)$$

To eliminate the normalization constants A and B , we can divide the first equation by the second :

$$\frac{k j_l(\kappa R)}{\kappa j_l'(\kappa R)} = \frac{\cos(\delta_l) j_l(kR) - \sin(\delta_l) y_l(kR)}{\cos(\delta_l) j_l'(kR) - \sin(\delta_l) y_l'(kR)} \quad (23)$$

We can isolate the scattering phase shift :

$$\tan(\delta_l) = \frac{k j_l'(kR) - \kappa j_l'(\kappa R) \alpha}{\kappa y_l'(kR) - \kappa y_l(kR) \alpha} \quad (24)$$

with $\alpha \equiv j_l'(\kappa R)/j_l(\kappa R)$.

As $|V_0| \ll E$ and $kR \ll 1$, we also have $\kappa R \ll 1$. Note that we cannot simply use the asymptotic forms of j_l and y_l proved in the first exercise : the derivatives are mixing different orders. This is why we use the identity showed in the hint. Hence $j_l(x) \approx \frac{x^l}{(2l+1)!!}$ for $x \ll 1$, so $j_{l+1}(\kappa R)/j_l(\kappa R) = \frac{\kappa R}{2l+3}$ to leading order. We also know that $y_l(x) = -\frac{(2l-1)!!}{x^{l+1}}$ for $x \ll 1$. Therefore the phase shift becomes

$$\begin{aligned} \tan \delta_l &= \frac{(\kappa R)^2 j_l(kR)/(2l+3) - kR j_{l+1}(kR)}{(\kappa R)^2 y_l(kR)/(2l+3) - kR y_{l+1}(kR)} \\ &= \frac{(\kappa R)^2 (kR)^l / [(2l+3)!!] - (kR)^{l+2} / [(2l+3)!!]}{- (2l-1)!! (\kappa R)^2 / [(2l+3)(kR)^{l+1}] + (2l+1)!! / (kR)^{l+1}} \\ &\approx (kR)^{2l+1} \frac{(\kappa R)^2 - (kR)^2}{(2l+3)!! (2l+1)!!} \\ &= \frac{(kR)^{2l+3}}{(2l+3)!! (2l+1)!!} \left[\frac{\kappa^2}{k^2} - 1 \right]. \end{aligned} \quad (25)$$

where we ignore the first term in the denominator for $kR \ll 1$. Clearly $l = 0$ dominates, and

$$\tan \delta_0 = \frac{1}{3} (kR)^3 \left[\frac{E - V_0}{E} - 1 \right] = -\frac{1}{3} (kR)^3 \frac{V_0}{E} = -\frac{1}{3} k \frac{2mV_0 R^3}{\hbar^2} \approx \delta_0 \approx \sin \delta_0. \quad (26)$$

The total cross section is

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sin^2 \delta_0 = \frac{16\pi m^2 V_0^2 R^6}{9 \hbar^4}. \quad (27)$$

2. The next most important term is the p -wave. The phase shift is

$$\tan \delta_1 = -\frac{1}{45}(kR)^5 \frac{V_0}{E} \approx \delta_1 \approx \sin \delta_1. \quad (28)$$

With just s - and p -waves the differential cross section is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{k^2} \left| e^{i\delta_0} \sin \delta_0 + 3e^{i\delta_1} \sin \delta_1 \cos \theta \right|^2 \\ &\approx \frac{1}{k^2} \left(\sin^2 \delta_0 + 6 \cos(\delta_0 - \delta_1) \sin \delta_0 \sin \delta_1 \cos \theta \right). \end{aligned} \quad (29)$$

which is of the form $\frac{d\sigma}{d\Omega} = A + B \cos \theta$. Since $\delta_1 \ll \delta_0 \ll 1$ we have $\cos(\delta_0 - \delta_1) \approx 1$ and

$$\frac{B}{A} = \frac{6 \sin \delta_1}{\sin \delta_0} = 6 \cdot \frac{3}{45}(kR)^2 = \frac{2}{5}(kR)^2. \quad (30)$$

5. Scattering off $\frac{1}{r^2}$ potential

1. One has to solve the radial Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \left(\frac{l(l+1) + A}{r^2} \right) \right] u_l = E u_l. \quad (31)$$

For $A = 0$, the known solution involves spherical Bessel functions $j_l(kr)$ and Neumann functions $y_l(kr)$:

$$R_l = \frac{u_l}{r} = a_l j_l(kr) + b_l y_l(kr), \quad (32)$$

with $b_l = 0$ to satisfy the boundary condition $R_l(r \rightarrow 0) < \infty$. For $A \neq 0$, we have:

$$R_l = a_l j_n(kr), \quad (33)$$

where n is defined by the quadratic equation $n(n+1) = l(l+1) + A$.

2. The asymptotic form of R_l for large r is:

$$R_l(r) \sim j_n(kr) \sim \frac{1}{r} \sin\left(kr - \frac{n\pi}{2}\right) \quad (34)$$

We want to compare this with the definition of the scattering phase shift:

$$R_l(r) \sim \frac{1}{r} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right) \quad (35)$$

Solving for n yields:

$$n = \frac{-1 \pm \sqrt{1 + 4(l(l+1) + A)}}{2}. \quad (36)$$

Since we want to let $n = l$ for $A \rightarrow 0$, we take the "+" part here. For small A , we use Taylor expansion to find $n \approx l + \frac{A}{2l+1}$.

The phase shift δ_l for small A is approximately :

$$\delta_l = -\frac{\pi A}{2(2l+1)}. \quad (37)$$

The total cross section σ is given by :

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l \approx \frac{\pi^3 A^2}{k^2} \sum_{l=0}^{\infty} \frac{1}{2l+1} \rightarrow \infty. \quad (38)$$

The series of inverses of odd integers diverges, hence the total scattering cross section also diverges. This happens because the $1/r^2$ potential is too long range : it does not decay fast enough as $r \rightarrow \infty$. We need at least a cubic decay $V \sim 1/r^3$ to have a finite cross section.