QUANTUM PHYSICS III

Problem Set 9 11 November 2025

1. Optical theorem in the Born approximation

The optical theorem is a precise statement about the relation between the scattering amplitude in the forward direction $f(\mathbf{p} \to \mathbf{p})$ and the total cross section of the scattering process,

$$\sigma = \frac{4\pi}{p} \mathrm{Im}\, f(\mathbf{p} \to \mathbf{p}) \ . \tag{1}$$

Suppose that the Hamiltonian $\hat{H}$ of the scattering system is of the form that permits the usage of the Born approximation. In particular, $\hat{H} = \hat{H}_0 + \lambda \hat{V}$ with $\hat{H}_0$ the free Hamiltonian and λ the small parameter. Then, the both sides of eq. (1) can be written as series with respect to λ, and the terms with the same power of λ must coincide.

1. Check explicitly the relation (1) to the first nontrivial order in λ.
   Hint : Note that λ enters the total cross section at least quadratically.

2. Scattering amplitude in a spherically-symmetric potential

Consider the particle of mass m and momentum p moving in a spherically-symmetric potential $V(r)$.

1. Show that in the first Born approximation the scattering amplitude is written as

$$f(\mathbf{p} \to \mathbf{p}') = -\frac{2m}{q} \int_0^\infty r V(r) \sin(qr) dr \ , \tag{2}$$

   where q is the momentum transfer

$$q = 2p \sin \frac{\theta}{2} \ . \tag{3}$$

2. Using eq. (2), compute the scattering amplitude and the differential cross section of the particle in the repulsive potential

$$V(r) = V_0 e^{-r^2/a^2} \ . \tag{4}$$

3. Scattering in a square-well potential

A nonrelativistic particle of mass m and momentum p is scattered by the following potential,

$$V(r) = \begin{cases} -V_0 \ , & r < R \ , \\ 0 \ , & r > R \ . \end{cases} \tag{5}$$

1. In the first Born approximation, find the differential cross section of the scattering in the potential (5). Plot it schematically, indicating angular units.
2. How to extract the value of R from the measured angular distribution of the scattered particles ?
3. Assuming the scattered particles to be protons and $R = 5 \cdot 10^{-13}\ cm$, roughly how high must the energy of the particles be in order for the scattering to be sensitive to R ?
4. Compute the total cross section σ of the scattering in the potential (5).
   Hint : Rewrite the solid angle element $d\Omega$ through dq with q the momentum transfer.
5. Find σ in the limit of slow scattering, i.e., when the wave length of the particles is much larger than the size of the potential. Explain the observed dependence of σ on the initial momentum p in this limit.
6. Find σ in the opposite limit of fast scattering. What is the dependence on p in this case ?

4. Cauchy's theorem and the completeness relation

Let $\hat{G}(z) = \dfrac{1}{z - \hat{H}}$, where $\hat{H}$ is a Hamiltonian containing a finite amount of bound states $|n\rangle$ with energies $E_n < 0$ and a continuous spectrum $|p\rangle$ beginning at $E = 0$.

1. Using the completeness relation for the eigenstates of $\hat{H}$,

$$\sum_n |n\rangle\langle n| + \int dp |p\rangle\langle p| = 1 \ , \tag{6}$$

   prove that

$$\oint_C \hat{G}(z) dz = 2\pi i \sum_n |n\rangle\langle n| \ , \tag{7}$$

   where the contour C is shown in figure 1.

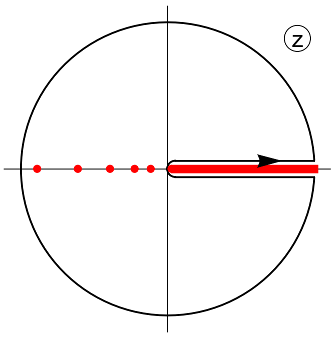

FIG. 1 – The contour of integration C. Red dots mark the bound states of $\hat{H}$, the red line represents the branch cut corresponding to the continuous spectrum.

5^*. The nucleus form factor

The study of the scattering of high-energy electrons from nuclei has yielded much interesting information about the charge distributions in nuclei and nucleons. In this exercise we neglect the electron spin and assume that the nucleus remains fixed in space. Let $\rho(\mathbf{x})$ denote the charge density in the nucleus.

Let $f_0(\mathbf{p} \to \mathbf{p}')$ be the scattering amplitude in the first Born approximation for the scattering of an electron from a point nucleus of charge Ze. Let $f(\mathbf{p} \to \mathbf{p}')$ be the scattering amplitude, also in the first Born approximation, for the scattering of an electron from a real nucleus of the same charge. The quantity $F = F(\mathbf{q}^2)$ defined as

$$f(\mathbf{p} \to \mathbf{p}') = F(\mathbf{q}^2) f_0(\mathbf{p} \to \mathbf{p}') \tag{8}$$

is called the form factor (it is easily seen that F in fact depends on $\mathbf{p}$ and $\mathbf{p}'$ only through the momentum transfer).

1. Relate the form factor $F(\mathbf{q}^2)$ to the Fourier transform of the charge density $\rho(\mathbf{x})$.
   Indication : Consider the case of the spherically symmetric distribution of charge in the nucleus.
2. Figure 2 shows some experimental results pertaining to the form factor of the proton. Based on these data, compute the mean-square radius of the proton.
   Hint : Find and use the relation between the mean-square radius and derivative of $F(\mathbf{q}^2)$ with respect to $\mathbf{q}^2$ at $\mathbf{q}^2 = 0$.

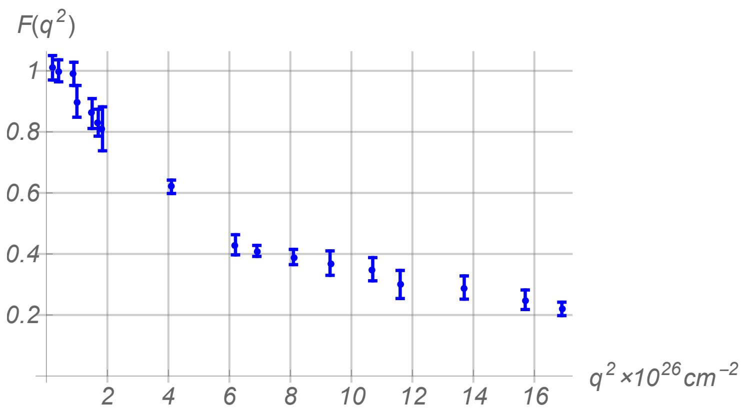

FIG. 2 – The measured form factor as a function of the momentum transfer.