QUANTUM PHYSICS III

Problem Set 8 4 November 2025

1. On integrals involving the delta-function

1. Compute the integral
$$\int_{-\infty}^{\infty} dx\, f(x)\delta(ax^2 + bx + c) \tag{1}$$
with $f(x)$ the real function and a, b, c the reals.
2. Compute the radial part of the integral
$$\int d^3\mathbf{p}\, f(\mathbf{p})\delta(E_{p'} - E_p) \tag{2}$$
with f the real function of the momentum $\mathbf{p}$.

2. Free particle's Green function in three dimensions

Recall that the Green function for the free Hamiltonian $\hat{H}_0$ is written as
$$\hat{G}_0(z) = \frac{1}{z - \hat{H}_0}, \quad z = x + i\epsilon\,. \tag{3}$$

1. Show that
$$\langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{z - E_p}\,, \quad E_p = \frac{p^2}{2m}\,, \tag{4}$$
where we put $\hbar = 1$.
2. Compute the integral above to yield
$$\langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle = -\frac{m}{2\pi}\frac{e^{i\sqrt{2mz}\,|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{5}$$

   Hint : First, compute the angular part of the integral. After that, close the contour of integration and use the method of residues. Note that the prescription $z = x + i\epsilon$, $\epsilon > 0$ plays a crucial role in this calculation.
3. Check that the result (5) is indeed the Green function of the stationary Schrodinger equation, that is
$$\langle \mathbf{x}|(z - \hat{H}_0)\hat{G}_0(z)|\mathbf{x}'\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,. \tag{6}$$
4. Calculate the difference
$$\langle \mathbf{x}|\hat{G}_0(E + i\epsilon) - \hat{G}_0(E - i\epsilon)|\mathbf{x}'\rangle \tag{7}$$
between the values of $\hat{G}_0(z)$ on the upper and the lower sides of the branch cut at $E > 0$.

5. Find an asymptotic behavior of the Green function $\langle \mathbf{x}|\hat{G}_0(z)|\mathbf{x}'\rangle$ in the limit of large separation of points (say, send $|\mathbf{x}| \to \infty$ while keeping $|\mathbf{x}'|$ finite).

3. Friedel sum rule

Let $N(E) = \sum_n \delta(E - E_n)$ be the density of eigenstates of the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, and $N_0(E)$ the density for the free Hamiltonian $\hat{H}_0$.

1. Express $N(E)$ through the Green function $\hat{G}$ of $\hat{H}$.
   Hint : Use the relation
   $$\frac{1}{x + i\epsilon} = -i\pi\delta(x) + \mathcal{P}\frac{1}{x} , \tag{8}$$
   where $\mathcal{P}$ denotes the principal value.
2. Deduce the following sum rule,
   $$N(E) - N_0(E) = \frac{1}{\pi}\frac{d}{dE}\arg\det\left(\hat{G}(E + i\epsilon)\hat{G}_0^{-1}(E + i\epsilon)\right) , \tag{9}$$
   where $\hat{G}_0$ is the Green function of the free Hamiltonian.

4. Slow scattering in a gas

Consider a non-relativistic particle of momentum p, scattering off the potential with the characteristic range R. We say that the particle is slow if

$$p \lesssim \frac{\hbar}{R} . \tag{10}$$

Considering this regime is important since, as we will see later, the scattering amplitudes can behave qualitatively different depending on whether the condition (10) fulfills or not.

1. The range of the potential between two hydrogen atoms is approximately 4 Å. For a gas in thermal equilibrium, find a numerical estimate of the temperature below which the atom-atom scattering is essentially slow.