QUANTUM PHYSICS III

Problem Set 7 28 October 2025

1. Interaction picture

Consider a system with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is the free Hamiltonian and $\hat{V}$ is the interaction. Define the interaction picture for states and operators via the relations

$$\begin{aligned} \Psi_I(t) &= \hat{U}_0^\dagger(t)\Psi_S(t) , \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t)\hat{A}_S\hat{U}_0(t) , \end{aligned} \tag{1}$$

where $\hat{U}_0(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}$, and the subscript S denotes quantities in the Schrodinger picture.

1. Find the relation between the states and operators in the interaction and Heisenberg pictures.
2. Show that the evolution of the wave function in the interaction picture is described by the interaction term $\hat{V}$ in the same picture, i.e.

$$-\frac{\hbar}{i}\frac{d}{dt}\Psi_I(t) = \hat{V}_I\Psi_I(t) . \tag{2}$$

3. Express the evolution operator in the interaction picture $\hat{U}_I(t)$ through $\hat{U}(t)$ and $\hat{U}_0(t)$. Find a differential equation which $\hat{U}_I(t)$ obeys and determine the initial condition for it.

2. Unitarity versus isometry

Recall that the operator $\hat{U}$ acting in the Hilbert space $\mathcal{H}$ is called unitary if

$$\mathcal{D}(\hat{U}) = \mathcal{H} , \quad \mathcal{R}(\hat{U}) = \mathcal{H} , \quad \hat{U}^\dagger\hat{U} = 1 , \tag{3}$$

where the last equality should be understood in the operator sense,

$$\langle\Phi|\hat{U}^\dagger\hat{U}|\Phi\rangle = \langle\Phi|\Phi\rangle = 1 , \quad \forall\Phi \in \mathcal{H} . \tag{4}$$

1. Prove that the set of conditions (3) is equivalent to the following set,

$$\mathcal{D}(\hat{U}) = \mathcal{H} , \quad \hat{U}^\dagger\hat{U} = 1 , \quad \hat{U}\hat{U}^\dagger = 1 . \tag{5}$$

2. Prove that if $\mathcal{H}$ is finite-dimensional, the conditions (5) can be eased to

$$\mathcal{D}(\hat{U}) = \mathcal{H} , \quad \hat{U}^\dagger\hat{U} = 1 . \tag{6}$$

3. Assuming $\mathcal{H}$ to be infinite-dimensional and with the basis $|1\rangle, |2\rangle, ..., |n\rangle, ...$, construct the sequence of unitary operators $\hat{U}(\lambda)$ such that $\lim_{\lambda\to 0} \hat{U}(\lambda) = \hat{\Omega}$, where $\hat{\Omega}$ is an isometric non-unitary operator.

3. Semiclassical S-matrix in one dimension

Consider a one-dimensional potential barrier located around the point $x = 0$. Assume that at large $|x|$, the potential falls off fast enough to ensure the plane wave asymptotic solutions of the Schroedinger equation. Let $|\Psi_{in}\rangle$ be the state representing a localized right-moving wave packet at large negative x. We are interested in how this wave packet transforms as it scatters off the barrier. Denote by $|\Psi_{out}\rangle$ the state representing the packet transmitted through the barrier in the region of large positive x. Then, one can define the operator $\hat{S}$ such that

$$\hat{S}\,|\Psi_{in}\rangle = |\Psi_{out}\rangle\ . \tag{7}$$

1. With the transmission coefficient of the barrier given by

$$D(q) = 1 - e^{-q^2/q_0^2}\ , \tag{8}$$

compute the matrix elements of $\hat{S}$ in the space of Gaussian functions,

$$S(p', \sigma'; p, \sigma) \equiv \langle \Psi_{p',\sigma'}|\hat{S}\,|\Psi_{p,\sigma}\rangle\ , \quad \Psi_{p,\sigma}(x,t) = C_{p,\sigma} e^{-\frac{\left(x-\frac{p}{m}t\right)^2}{4\sigma^2}}\ , \tag{9}$$

where $C_{p,\sigma}$ is the appropriate normalization constant (see Problem 2 of Problem set 1).