QUANTUM PHYSICS III

Problem Set 5 7 October 2025

1. Perturbation of potential and WKB

Consider a small time-independent perturbation $\delta V(x)$ of the potential $V(x)$. How does this perturbation affect the semiclassical energy levels of a particle in this potential ? Introduce the momenta of the particle in the non-perturbed and perturbed potentials,

$$p_0(x) = \sqrt{2m(E_n - V(x))} \ , \quad p(x) = \sqrt{2m(E_n + \delta E_n - V(x) - \delta V(x))} \ . \tag{1}$$

1. By comparing the quantization conditions for $p_0(x)$ and $p(x)$, find the expression for δE_n in terms of $\delta V(x)$ and $p_0(x)$.
2. Reduce this expression to the form

$$\delta E_n = \frac{1}{T_n} \int_0^{T_n} dt \, \delta V[x_n(t)] \ , \tag{2}$$

where T_n denotes the period of oscillations of the particle on the n'th energy level, and $x_n(t)$ is particle's classical trajectory.

2. Rosen-Morse potential

Recall that one remarkable property of the harmonic oscillator is that its WKB energy spectrum *coincides* with the exact one, despite the formal inapplicability of the LO WKB approach to the energy levels with small n. Here we consider the so-called Rosen-Morse potential (see figure 1),

$$V(x) = -\frac{V_0}{\cosh^2(\frac{x}{x_0})} \ , \tag{3}$$

which possesses the same property. Like the harmonic oscillator, this potential is exactly soluble, and the comparison of the exact answer with the result of the LO WKB reveals their coincidence. To make the result more illustrative, let us set

$$x_0 = 2m = \hbar = 1 \ , \quad V_0 = \frac{49}{4} \ . \tag{4}$$

1. Find the WKB spectrum of the particle in the potential (3), subject to notations (4). *Hint :* Computation of the integral $J = \int p dx$ by hand is complicated. Compute instead dJ/dE and then integrate the result. Use the following change of variables,

$$y = \sinh x \ . \tag{5}$$

2. Find for which n you can trust the results according to the LO WKB applicability conditions.

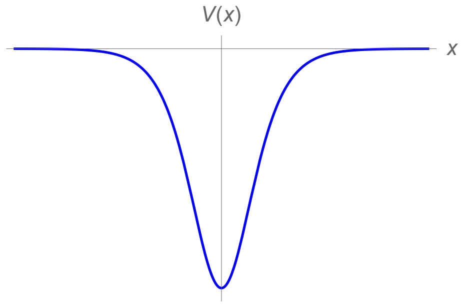

FIG. 1 – The Rosen-Morse potential

3. Tunneling through a parabolic barrier

Consider the particle with energy E, moving from $x \to -\infty$ towards the potential of the form

$$V(x) = \begin{cases} V_0 \left(1 - \dfrac{x^2}{x_0^2}\right) , & |x| < x_0 \\ 0 , & |x| > x_0 \ . \end{cases} \tag{6}$$

1. Assuming $E < V_0$, find the transmission coefficient of the potential (6).
2. For which E you can trust the result of the previous calculation ?

4. Lifetime in a cubic potential

Consider the nonrelativistic particle with mass m and energy E, confined in the well of the potential $V(x) = V_0 x^2 (x_0 - x)$, with $V_0 > 0$, $x_0 > 0$.

1. Assuming $E \ll V_0$, find the lifetime of the particle in the well.

5*. Super-WKB approach

In a variety of problems in Quantum mechanics it is useful to look for the following decomposition of the potential $V = V(x)$,

$$V = W^2 - \frac{\hbar}{\sqrt{2m}} W' \ . \tag{7}$$

The function W is called the *superpotential* associated with the potential V. If the superpotential is known, one can build a perturbation theory in $\hbar$ starting from W^2 instead of V. This corresponds to some rearrangement in the series of the original WKB theory. It turns out that for many classes of potentials such rearrangement improves significantly the predictions of the standard WKB.

1. Using eq. (7), rewrite the Bohr-Sommerfeld quantization rule in terms of the superpotential and to the LO in $\hbar$.
   Indication : Consider the case when on the turning points, the superpotential takes either opposite ("unbroken SUSY") or equal ("broken SUSY") values.

2. Assuming the oppositve values of the superpotential on the turning points, find the ground state energy E_0.
3. Using the quantization rule found in p.1, compute the energy spectrum of a particle in the "inverse hydrogen atom" potential

$$V(x) = -\frac{1}{x} + \frac{x(x+2)}{(1+x+\frac{1}{2}x^2)^2} + \frac{1}{16} \,, \tag{8}$$

where x is a dimensionless variable, and we put $2m = \hbar = 1$ for simplicity. Compare the answer with the result of the standard LO WKB approximation, and with the exact formula

$$E_n = \frac{1}{16}\frac{n(n+4)}{(n+2)^2} \,, \quad n = 0, 1, 2, ... \tag{9}$$

Hint : The superpotential associated with the potential (8) is given by

$$W(x) = \frac{x^6 - 16x^4 - 56x^3 - 108x^2 - 240x - 192}{4x(x^2+2x+2)(x^3+6x^2+18x+24)} \,. \tag{10}$$

6*. Scattering off the peak

From the lecture notes we know how to find the transmission and reflection coefficients D and R in the LO WKB approximation and in the case when the energy of an incident particle E is below the height of the barrier V_0. What happens in the limit $E \to V_0$? In this limit, we have only one turning point x_0 of even multiplicity k, such that $V(x_0) = V_0$. Then, in order to match the WKB wave functions to the both sides of the turning point, one has to expand $V(x)$ up to the k'th order around that point.
Let us take for simplicity $x_0 = V_0 = 0$, $k = 2$, and let $V(x)$ approach some constant negative value at $x \to \pm\infty$. Consider the particle with mass m and energy $E < 0$, moving towards x_0 from $x \to -\infty$.

1. Find the region $\mathcal{R}$ of x where it is legitimate both
   (a) to approximate the potential by the first nonzero term of its Taylor expansion around the turning point, and
   (b) to approximate the momentum p by the first two terms of its Taylor expansion in $|E/V(x)|$.
2. Using the approximations found in p.1, write the LO WKB wave functions of the particle in the region $\mathcal{R}$, to the left and to the right sides from the turning point.
3. Now use a path in the region $\mathcal{R}$ continued to the complex plane to connect the LO WKB wave functions found in p.2. Extract the coefficients D and R.
4. Investigate the limit of D and R as $E \to 0$.