QUANTUM PHYSICS III

Problem Set 13 9 December 2025

1. Non-relativistic limit of the Dirac equation

Previously we discussed the Dirac equation for the particle moving in an external electromagnetic field. Let us now investigate the non-relativistic limit of this equation which we expect to be of the form of the Schroedinger equation, and the form of the leading-order relativistic corrections. To simplify the treatment, consider the spherically-symmetric static electric field for which

$$\vec{A} = 0 \,, \quad e\Phi = V(r) \,. \tag{1}$$

Then, the Dirac equation reads as follows,

$$H\Psi = \mathcal{E}\Psi \,, \quad H = c\vec{\alpha} \cdot \vec{p} + \beta mc^2 + V(r) \,, \quad \mathcal{E} = E + mc^2 \,, \tag{2}$$

where $\Psi = (\phi, \chi)^T$ is the Dirac spinor and we put $\hbar = 1$. The non-relativistic limit implies

$$\frac{p^2}{2m} \ll mc^2 \,, \quad V(r) \ll mc^2 \,, \tag{3}$$

hence the expansion in eq. (2) can be performed with respect to the speed of light c. We will work in the Dirac representation of the Dirac matrices.

1. Find the expression for χ through ϕ with the accuracy $O(1/c^3)$. Using it, obtain an equation on ϕ with the accuracy $O(1/c^2)$.

2. Rewrite the equation for ϕ in the form

$$E\phi + \frac{p^2}{2m}\frac{E}{2mc^2}\phi = \tilde{H}\phi \tag{4}$$

with $\tilde{H}$ some hermitian operator. To simplify the l.h.s. of eq. (4), introduce a new variable

$$\xi = \sqrt{1 + \frac{p^2}{2m}\frac{1}{2mc^2}}\phi \tag{5}$$

and rewrite eq. (4) in the form of the Schroedinger equation

$$H_{eff}\xi = E\xi \,. \tag{6}$$

3. Extract the part of H_{eff} which does not depend on c. It should give you the usual Schroedinger equation on ξ.

4. Now extract the $O(c^{-2})$-term in H_{eff} and bring it to the form

$$\underbrace{-\frac{1}{2mc^2}\frac{(p^2)^2}{4m^2}}_{\text{rel. correction to the energy}} + \underbrace{\frac{\hbar\vec{\sigma}}{2m \cdot 2mc^2} \cdot \left(\vec{\nabla}V \times \vec{p}\right)}_{\text{spin-orbital interaction}} + \underbrace{\frac{\hbar^2}{8m^2c^2}\Delta V}_{\text{the Darwin term}} \,. \tag{7}$$

5. Rewrite the expression (7) for the Coulomb potential

$$V(r) = -\frac{Ze^2}{r} \ . \tag{8}$$

Indication: recall that the orbital momentum of the particle is $\vec{L} = [\vec{x}, \vec{p}]$, and the vector of its spin is $\vec{s} = \hbar\vec{\sigma}/2$.

2. Zitterbewegung

The physical meaning of the first two relativistic corrections in eq. (7) is quite clear: the first comes from the expansion of the relativistic energy of the particle, $E^2 = m^2c^4 + (p^2/2m)^2$, and the second represents the spin-orbital coupling. But what is the Darwin term? It can be attributed to a peculiar motion of a Dirac particle called *Zitterbewegung*. One way to see it is by using the Heisenberg equation of motion. Let us take the free particle Dirac Hamiltonian H_D, then

$$i\hbar\frac{dO}{dt} = [O, H_D] \ , \tag{9}$$

where O is some operator (an observable).

1. By taking $O = \ddot{\vec{x}}$ in eq. (9), obtain an equation on the observable $\dot{\vec{x}}$. Find the general solution of this equation. Integrating it, obtain an expression for the coordinate $\vec{x}$.
2. Determine all arbitrary constants in the expression for $\vec{x}$ found above by comparing its derivatives with the Heisenberg equations written for $\vec{x}$ and $\dot{\vec{x}}$.
3. What is the behavior of the coordinate $\vec{x}$? Does it oscillate and if so, what is the frequency of the oscillations (the formula for it and the numerical value)?
4. If the Dirac particle is moving in a (weak) external electric potential V, the average value of the potential felt by the particle is given by

$$\langle V \rangle = \frac{1}{2}\langle x^i x^j \rangle \frac{\partial^2 V}{\partial x^i \partial x^j} \ . \tag{10}$$

   Making use of the explicit formula for $\vec{x}$, find $\langle V \rangle$. Compare with the third term of eq. (7).
   Hint: What is the value of $\langle x^i x^j \rangle$ in the isotropic background?