QUANTUM PHYSICS III

Problem Set 12 2 December 2025

1. General solution of the Dirac equation

Due to linearity, the Dirac equation

$$-\frac{\hbar}{i}\frac{\partial \Psi_D}{\partial t} = H_D \Psi_D \tag{1}$$

admits a decomposition of its general solution to the plane-wave functions,

$$\Psi_D = e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - \omega_P t)} u_P \ , \tag{2}$$

where u_P is some function of the momentum $\mathbf{p}$.

1. Rewrite eq. (1) as an equation on u_P.
2. Find the necessary and sufficient condition for this equation to have a non-zero solution. What is the physical meaning of this condition?
3. Find the general solution of the equation above.
   Hint: At this point it is convenient to remember that u_P is a column $(\phi_P,\ \chi_P)^T$ of two-component functions ϕ_P and χ_P.
4. Rewrite the general solution in the non-relativistic limit $p \ll m$.

2. Properties of the Dirac matrices

Recall that the Dirac Hamiltonian H_D is given by the following 4×4 matrix,

$$H_D = \sum_{i=1}^{3} \alpha_i p_i + \beta m \ , \tag{3}$$

where

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} . \tag{4}$$

Here σ_i are the Pauli matrices, and I denotes the 2×2 identity matrix. The matrices α_i and β obey certain relations which, however, do not specify them fully, hence the choice (4) is not unique.

1. Given α_i, β, one can define new matrices α'_i, β' via

   $$\alpha'_i = U\alpha_i U^{-1} \ , \quad \beta' = U\beta U^{-1} \ , \tag{5}$$

   where U is a unitary but otherwise arbitrary 4×4 matrix. Show that α'_i, β' form an appropriate set of matrices provided that α_i, β do.

2. Find the matrix U that transforms β, given in eq. (4), into $\beta' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Find α'_i corresponding to this transformation. This choice of the Dirac matrices is called the Weyl representation.

3. Write the Dirac equation in the Weyl representation and in the notation $\Psi_D = (\phi,\ \chi)^T$. Take the limit $m = 0$, and check if the components ϕ and χ satisfy the Klein-Gordon equation. Find the solution of this equation for the massless particle propagating along the x-direction.

4. Find the representation of the Dirac matrices α', β' such that Im α'_i = Re $\beta' = 0$.

3. One useful relation

1. Show that

$$(\vec{\sigma} \cdot \vec{\pi})(\vec{\sigma} \cdot \vec{\pi}) = \vec{\pi}^2 - \frac{e\hbar}{c}\vec{\sigma} \cdot \vec{B}\,, \tag{6}$$

where $\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$, $\vec{B} = \text{rot}\,\vec{A}$, and $\vec{\sigma}$ denotes the triplet of the Pauli matrices.

4. On Landau levels

In this exercise we are interested in energy levels of an electron in a uniform magnetic field. To find them, one should proceed in the same way as in the non-relativistic case. Namely, we take an *Ansatz* for stationary states,

$$\Psi = e^{-\frac{i}{\hbar}Et}\begin{pmatrix}\phi \\ \chi\end{pmatrix}\,, \tag{7}$$

and plug it into the Dirac equation in the external field,

$$-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = \left(c\vec{\alpha}\cdot\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right) + \beta mc^2 + e\Phi\right)\Psi\,. \tag{8}$$

This gives an eigenvalue problem for E whose solution will provide us with the desired energy levels.
Specifically, let us align the magnetic field along z-direction,

$$\vec{B} = \begin{pmatrix}0 \\ 0 \\ \mathcal{B}\end{pmatrix}\,. \tag{9}$$

We will work in the Dirac representation of the matrices α_i, β studied in Lectures. For simplicity, we also put $\hbar = c = 1$.

1. Show that the magnetic field (9) is reproduced by the following combination of the potentials,

$$\vec{A} = -\begin{pmatrix} y\mathcal{B} \\ 0 \\ 0 \end{pmatrix} , \quad \Phi = 0 \,. \tag{10}$$

Is this choice of $\vec{A}$ and Φ unique?

2. Substituting the Ansatz (7) and the potentials (10) into eq. (8), obtain an equation on the component ϕ.

3. Next, assume the following form of the general solution of the equation above,

$$\phi = e^{i(p_x x + p_z z)} \left(c_1 \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} \right) , \tag{11}$$

with c_1, c_2, p_x, p_z arbitrary constants. Find equations on the functions $F_+(y)$ and $F_-(y)$. By changing variables, reduce them to the form

$$\left(\frac{d^2}{d\xi^2} - \xi^2 + \alpha_\pm \right) F_\pm(\xi) = 0 \,. \tag{12}$$

4. Eq. (12) is of Hermite's type. It admits solutions provided that $\alpha_\pm = 2n + 1$, $n = 0, 1, 2, ...$ Having this in mind, derive the formula for the electron energy levels. What is the degeneracy of the ground level $n = 0$? Of the first excited level $n = 1$?