QUANTUM PHYSICS III

Problem Set 11 25 November 2025

1. Properties of spherical Bessel functions

Recall the definitions of the spherical Bessel and von Neumann functions (for $l \in \mathbb{N}$) :

$$j_l(\rho) = (-1)^l \rho^l \left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^l \frac{\sin\rho}{\rho}, \tag{1}$$

$$y_l(\rho) = -(-1)^l \rho^l \left(\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^l \frac{\cos\rho}{\rho}. \tag{2}$$

1. Compute the asymptotic behavior of j_l and y_l for $\rho \to 0$:

$$j_l(\rho) \sim \frac{\rho^l}{(2l+1)!!}, \tag{3}$$

$$y_l(\rho) \sim \frac{(2l-1)!!}{\rho^{l+1}}, \tag{4}$$

   "!!" is the double factorial, defined as $n!! = n \cdot (n-2) \cdot (n-4) \cdot \ldots$, with the usual convention $0!! = 1$.

2. Compute the asymptotic behavior of j_l and y_l for $\rho \to \infty$:

$$j_l(\rho) \sim \frac{1}{\rho} \sin\left(\rho - l\frac{\pi}{2}\right), \tag{5}$$

$$y_l(\rho) \sim \frac{1}{\rho} \cos\left(\rho - l\frac{\pi}{2}\right). \tag{6}$$

2. Scattering phase shift in a Yukawa potential

For a Yukawa potential,

$$V(r) = \frac{V_0}{\mu r} e^{-\mu r} \tag{7}$$

it can be shown that in the first Born approximation, the scattering amplitude is given by

$$f(\mathbf{p}' \leftarrow \mathbf{p}) = -\frac{2mV_0}{\mu} \frac{1}{2p^2(1-\cos\theta) + \mu^2} \tag{8}$$

θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$, and $p = |\mathbf{p}| = |\mathbf{p}'|$; (we also consider $\hbar = 1$ as usual).

1. Obtain an expression for the scattering phase shift δ_l in terms of Legendre functions of the second kind :

$$Q_l(\zeta) = \frac{1}{2} \int_{-1}^{1} \frac{P_l(\zeta')}{\zeta - \zeta'} \mathrm{d}\zeta' \tag{9}$$

We will assume that $|\delta_l| \ll 1$.

Hint. Make use of the orthogonality of the Legendre polynomials :

$$\int_{-1}^{1} \mathrm{d}\zeta P_l(\zeta) P_{l'}(\zeta) = \frac{2\delta_{ll'}}{2l+1} \tag{10}$$

2. Use the expansion formula :

$$Q_l(\zeta) = \sum_{n=0}^{\infty} \frac{(l+2n)!}{(2l+2n+1)!!(2n)!!} \cdot \frac{1}{\zeta^{l+2n+1}} \tag{11}$$

to show the following :

   (a) δ_l is negative, resp. positive, when the potential is repulsive, resp. attractive.

   (b) When the de Broglie wavelength is much longer than the range of the potential, δ_l is proportional to p^{2l+1}. Find the proportionality constant.

3. Scattering off a spherical potential

Consider the potential $V(r)$ given by

$$V(r) = \begin{cases} 0 & \text{for } r > R, \\ \infty & \text{for } r < R. \end{cases} \tag{12}$$

1. Derive an expression for the s-wave ($l = 0$) phase shift. By obtaining a general expression for the scattering phase shift δ_l in the limit $kR \ll 1$, justify contributions beyond s-wave can be neglected.
2. What is the total cross section $\sigma = \int (\frac{d\sigma}{d\Omega}) d\Omega$ in the extreme low-energy limit $k \to 0$? Compare your answer with the geometric cross section πa^2. Use the following relations :

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \tag{13}$$

$$f(\theta) = \left(\frac{1}{k}\right) \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos\theta). \tag{14}$$

4. Scattering off a constant potential

Consider a potential

$$V = \begin{cases} 0 & \text{for } r > R, \\ V_0 = \text{constant} & \text{for } r < R, \end{cases} \tag{15}$$

where V_0 may be positive or negative.

1. Using the method of partial waves, show that for $|V_0| \ll E = \frac{\hbar^2 k^2}{2m}$ and $kR \ll 1$, the scattering phase shift can be written as :

$$\tan \delta_l = \frac{(kR)^{2l+3}}{(2l+3)!!(2l+1)!!} \left[\frac{\kappa^2}{k^2} - 1 \right], \tag{16}$$

where $E - V_0 \equiv \frac{\hbar^2 \kappa^2}{2m}$.

Furthermore, show that the differential cross section is isotropic and that the total cross section is given by

$$\sigma_{\text{tot}} = \left(\frac{16\pi}{9} \right) \left(\frac{m^2 V_0^2 R^6}{\hbar^4} \right). \tag{17}$$

Hint : the following identity may be useful :

$$f_l'(x) = \frac{l}{x} f_l(x) - f_{l+1}(x), \tag{18}$$

for f any spherical Bessel or von Neumann function.

2. Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B \cos\theta. \tag{19}$$

Obtain an approximate expression for B/A.

5. Scattering off a $\frac{1}{r^2}$ potential

Let ($A > 0$) and consider the potential

$$V(r) = \frac{\hbar^2 A}{2mr^2}. \tag{20}$$

1. Write down the eigenvalue equation of the Hamiltonian associated with this potential, and find a solution in terms of spherical Bessel functions.

   Hint : Even though the spherical Bessel functions $j_l(x)$ were defined for an integer l, they can also be extended to any complex l.

2. Obtain the phase shifts exactly. Show that for $A \ll 1$, one approximately has

$$\delta_l = -\frac{\pi}{2} \frac{A}{2l+1}. \tag{21}$$

What is the value of the total cross section ?