QUANTUM PHYSICS III

Problem Set 10 18 November 2025

1. Applicability condition of the first Born approximation

On physical grounds, one can expect the accuracy of the Born approximation to increase when the energy of the scattered particles becomes higher or when the interaction term in the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ becomes weaker. To make this reasoning quantitative, we say that the (first) Born approximation works well if the difference between the asymptotic wave function $\Psi_{in}(\mathbf{x}) = \langle \mathbf{x}|\Psi_{in}\rangle$ and the wave function $\Psi_0(\mathbf{x}) = \langle \mathbf{x}|\Psi_0\rangle = \langle \mathbf{x}|\hat{\Omega}_+|\Psi_{in}\rangle$ is small, in particular

$$|\Psi_{in}(\mathbf{0}) - \Psi_0(\mathbf{0})| \ll |\Psi_{in}(\mathbf{0})| \ . \tag{1}$$

1. Show that for a spherically symmetric potential $V(r)$ the condition (1) implies

   $$\frac{m}{p}\left|\int_0^\infty dr\ V(r)(1 - e^{2ipr})\right| \ll 1 \ , \tag{2}$$

   where p is the momentum of the particle and m is its mass.

2. As a model example, consider the square well potential,

   $$V(r) = \begin{cases} -V_0 \ , & r < R \ , \\ 0 \ , & r > R \ . \end{cases} \tag{3}$$

   Substitute this potential into eq. (2) and obtain the algebraic inequality the quantities m, V_0 and R must satisfy in the limit of slow scattering, $pR \ll 1$.

3. Show that this inequality can be rewritten as

   $$\sigma \ll 4\pi R^2 \ . \tag{4}$$

4. Work out the applicability condition for the potential (3) in the regime of fast scattering, $pR \gg 1$. Is it stronger or weaker than in the slow scattering limit?

5. Use eq. (2) to derive the applicability conditions for the Yukawa potential,

   $$V(r) = \frac{\alpha}{r}e^{-\mu r} \ , \tag{5}$$

   in the cases of slow ($p \ll \mu$) and fast ($p \gg \mu$) particles.

2. Towards the inverse scattering problem

Elastic scattering from some central potential $V(r)$ can be adequately calculated using the first Born approximation. Experimental results give the following qualitative behaviour of the scattering amplitude as a function of the momentum transfer q (see figure 1),

(i) For $q \lesssim q_0$, $|f(q)| \approx f_0$, $\dfrac{|f'(q)|}{q} \approx C$;

(ii) For $q \gtrsim q_0$, $|f(q)| \sim q^{-N/2}$, $N > 3$.

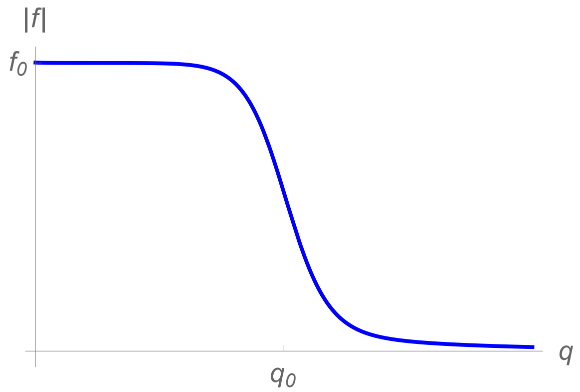

Fig. 1: The measured behaviour of the scattering amplitude.

1. What is the approximate size of the interaction region of the potential $V(r)$?
   Hint: Expand the expression for the scattering amplitude at small q by the powers of qr.

2. What is the behaviour of the potential at very small distances?

3. Truncation of the Coulomb potential

The scattering theory studied in this course is not directly applicable to many important physical situations, in particular, to the Coulomb scattering for which

$$V(r) = \frac{\alpha}{r} \; . \qquad (6)$$

It is easy to see that a straightforward attempt to compute the Born amplitude with the potential (6) results in the divergence. A possible way to produce meaningful results within the conventional scattering theory is to *truncate* the expression (6), i.e., to change its behaviour at large distances so that the scattering amplitude becomes well-defined. The answer obtained in this way makes sense as long as any measurement of physical observables with finite accuracy does not depend on a parameter of truncation and on a specific truncation procedure. If this is the case, the result is expected to be consistent with the one computed within the rigorous approach.

In this exercise, we consider two ways to improve the asymptotics of the Coulomb potential (6) at infinity: the exponential shielding

$$V_\rho(r) = \frac{\alpha}{r} e^{-r/\rho} \,, \tag{7}$$

and a sharp cutoff

$$V_\rho(r) = \begin{cases} \dfrac{\alpha}{r} \,, & r \leqslant \rho \,, \\ 0 \,, & r > \rho \,. \end{cases} \tag{8}$$

Here ρ is the truncation parameter which is assumed to be finite but arbitrarily large.

1. In the first Born approximation, compute the scattering amplitude $f_1(\mathbf{p} \to \mathbf{p}')$ for the potential (7). For which scattering angles does it have a well-defined limit at $\rho \to \infty$?

2. Compute, also in the first Born approximation, the amplitude $f_2(\mathbf{p} \to \mathbf{p}')$ for the potential (8). Does it have a limit at $\rho \to \infty$?

3. Show that by taking ρ sufficiently large, the ratio of the two answers $|f_2/f_1|$ can be made arbitrarily close to 1 in any experiment measuring the scattering angle θ with the finite accuracy $\Delta\theta$. In other words, prove that

$$\frac{1}{\Delta\theta} \int_\theta^{\theta+\Delta\theta} d\theta' \left| \frac{f_2(\theta')}{f_1(\theta')} \right| = 1 \,, \quad \theta \neq 0 \,, \quad \rho \gg \rho_0 \,. \tag{9}$$

   Find ρ_0 as a function of the initial momentum of the particle, the measured scattering angle θ and the systematic error $\Delta\theta$.

4. As another way to convince oneself in the legitimacy of the truncation procedure, consider the out wave packet produced by the potential (7) or (8),

$$\Psi_{out}(\mathbf{p}) = \Psi_{in}(\mathbf{p}) + \frac{i}{2\pi m} \int d^3\mathbf{p}' \; \delta(E_p - E_{p'}) f_{1,2}(\mathbf{p} \to \mathbf{p}') \Psi_{in}(\mathbf{p}') \,. \tag{10}$$

   Show that in the limit $q\rho \gg 1$ the difference between f_1 and f_2 makes no contribution to $\Psi_{out}(\mathbf{p})$.