Plasma Physics I

Solution to the Series 9 (November 15, 2025)

Prof. Christian Theiler

Swiss Plasma Center (SPC)
École Polytechnique Fédérale de Lausanne (EPFL)

Exercise 1

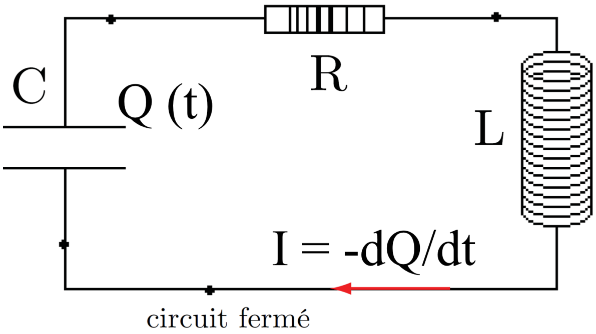

Figure 1: Circuit at $t = 0^+$.

a) At $t = 0^+$ (figure 1), the capacitor discharges (initial charge Q_0), a current I circulates and a voltage appears on each component:

$$V_R(t) = RI(t), \qquad V_L = L\frac{dI(t)}{dt}, \qquad V_C = -\frac{1}{C}\left[Q_0 - \int_0^t I(t')dt'\right] = -\frac{Q(t)}{C}. \tag{1}$$

From the Kirchhoff's law, the sum of the voltages around any closed circuit must be zero:

$$V_{\text{cycle}} = \sum_i V_i = 0, \qquad \forall\, t. \tag{2}$$

Using equation (2) on our circuit:

$$V_R + V_L + V_C = 0$$

and substituting the expression of the voltage on each component, eq.(1), we find:

$$L\ddot{Q}(t) + R\dot{Q}(t) + \frac{Q(t)}{C} = 0.$$

b) Introducing the quantities:

$$\delta = \frac{R}{2L}, \qquad \omega_{LC}^2 = \frac{1}{LC}, \tag{3}$$

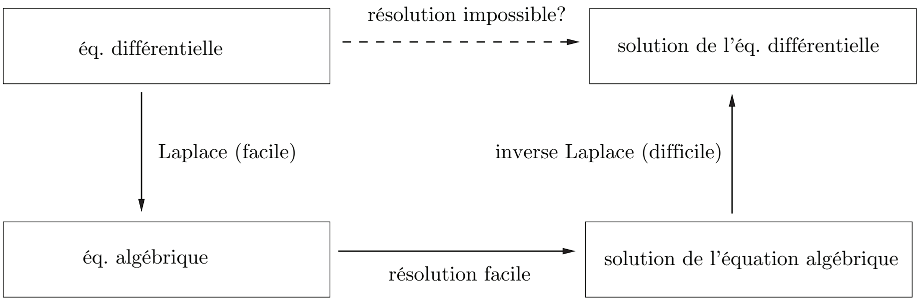

Figure 2: Solving differetial equations with Laplace.

we have:

$$\ddot{Q}(t) + 2\delta\dot{Q}(t) + \omega_{LC}^2 Q(t) = 0. \tag{4}$$

The solution of equation (4) is the temporal evolution of the charge in the capacitor. To solve it, we have two possibilities (see figure 2):

- Solve the system on the real-variables domain (position, time, ...)
- Use the Laplace transform to produce an easily solvable algebraic equation. From the solution in the Laplace domain, we can then find the general solution of the original differential equation using the inverse Laplace transform. The last step can be more complicated from the mathematic point of view.

To refresh the properties of the Laplace transform, we can solve the (4) using this method.

The Laplace transform:

$$\left(s^2 + 2\delta\, s + \omega_{LC}^2\right)\tilde{Q}(s) - (s + 2\delta)Q_0 - \dot{Q}_0 = 0 \tag{5}$$

and, using the initial conditions[1]:

$$\begin{cases} Q(0) = & Q_0 \\ \dot{Q}(0) = & 0 \end{cases} \tag{6}$$

we have:

$$\tilde{Q}(s) = \frac{(s + 2\delta)Q_0}{s^2 + 2\delta\, s + \omega_{LC}^2}. \tag{7}$$

c) The eq. (7) is the solution in the Laplace space. To obtain the solution in the real-space, we need to use the inverse Laplace transform:

$$Q(t) = \frac{1}{2\pi\, i}\int_{p_0 - i\infty}^{p_0 + i\infty} \tilde{Q}(s)\, e^{st} ds. \tag{8}$$

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[1] $\dot{Q} = 0$ because there is no current when $f < 0$.

Inverse Laplace transform

The *poles* of $\tilde{Q}(s)$ are:

$$s^2 + 2\delta\, s + \omega_{LC}^2 = 0 \qquad \Rightarrow \qquad s_{1,2} = -\delta \pm i\, \underbrace{\sqrt{\omega_{LC}^2 - \delta^2}}_{\omega_\gamma}$$

because $\omega_{LC}^2 - \delta^2 > 0$ and defining $\omega_0 = \delta + i\omega_\gamma$:

$$s^2 + 2\delta\, s + \omega_{LC}^2 = (s + \omega_0)(s + \omega_0^*), \tag{9}$$

therefore:

$$Q(t) = \frac{1}{2\pi\, i} \int_{p_0 - i\infty}^{p_0 + i\infty} \frac{(s + 2\delta)Q_0}{(s + \omega_0)(s + \omega_0^*)}\, e^{st} ds. \tag{10}$$

Notice that the poles have a negative real part (figure 3).

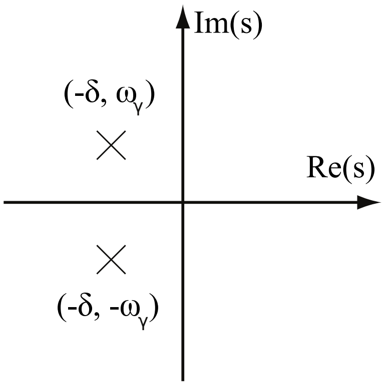

Figure 3: Poles of $\tilde{Q}(s)$ in the complex plane

To solve equation (10) we use the method of residues:

$$\oint_{\mathcal{C}} f(s) ds = 2\pi i \sum_i R_i, \tag{11}$$

where R_i are the residues of $f(s)$ in the region delimited by the contour $\mathcal{C}$.

We can choose our contour arbitrarily (Cauchy theorem, figure 4). It has to include the path $(p_0 - i\infty, p_0 + i\infty)$ and it has to be closed with a curve Γ. There are two possibilities:

(a) Path $\mathcal{C} = (p_0 - i\infty, p_0 + i\infty) + \Gamma_1$. The residues are not in the integration domain, therefore:

$$\oint_{\mathcal{C}} f(s) ds = 0 \quad \Rightarrow \quad \int_{p_0 - i\infty}^{p_0 + i\infty} f(s) ds = -\int_{\Gamma_1} f(s) ds.$$

We can use this method if the integral over Γ_1 can be easily computed.

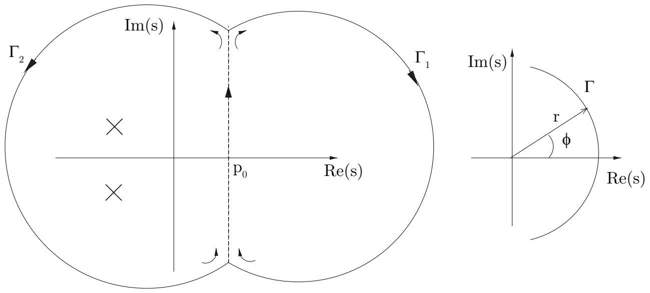

Figure 4: Left: Integration path in the complex plane. Right: Variable substitution to compute the integral in the complex plane.

(b) Path $\mathcal{C} = (p_0 - i\infty, p_0 + i\infty) + \Gamma_2$. We find:

$$\oint_{\mathcal{C}} f(s)ds = 2\pi i \sum_{i=1,2} R_i \qquad \Rightarrow \qquad \int_{p_0-i\infty}^{p_0+i\infty} f(s)ds = 2\pi i \sum_{i=1,2} R_i - \int_{\Gamma_2} f(s)ds$$

and, if we can show that:

$$\int_{\Gamma_2} f(s)ds = 0,$$

we can evaluate the integral (10) from the residues.

With the substitution (figure 4)

$$s = r\, e^{i\phi} \qquad \Rightarrow \qquad ds = ire^{i\phi}d\phi$$

and choosing $p_0 = 0$ (since $\mathrm{Re}\{\text{poles of } \tilde{Q}(s)\} < 0$), we have:

$$\int_{\Gamma_1,\Gamma_2} \tilde{Q}(s)\, e^{st}ds = i \int_{\phi_0}^{\phi_0+\pi} \tilde{Q}(s)\, re^{i\phi} \exp\big\{\big[re^{i\phi}\, t\big]\big\}d\phi.$$

If we take the norm of the integral:

$$\left|\int_{\phi_0}^{\phi_0+\pi} \tilde{Q}(s)e^{st}ds\right| = \left|i\int_{\phi_0}^{\phi_0+\pi} \tilde{Q}(s)re^{i\phi} \exp\big\{\big[re^{i\phi}\, t\big]\big\}d\phi\right| \leq$$

$$\leq \int_{\phi_0}^{\phi_0+\pi} \left|\tilde{Q}(s)\, r \exp\{[r(\cos\phi + i\sin\phi)\, t]\}\right| d\phi = \int_{\phi_0}^{\phi_0+\pi} \left|\tilde{Q}(s)\right| re^{rt\cos\phi}d\phi. \quad (12)$$

Changing the value of ϕ_0 we can consider both cases:

(a) $\phi_0 = -\pi/2$, $\phi \in [-\pi/2, \pi/2]$: we close the integration path on the right side of the integration domain.
The $\cos\phi$ term on the right side of equation (12) is positive (or zero) and, for $t > 0$:

$$\left|\int_{\Gamma_1} \tilde{Q}(s)\, e^{st}ds\right| \leq \lim_{r\to\infty} \int_{-\pi/2}^{\pi/2} \left|\tilde{Q}(s)\right| re^{rt|\cos\phi|}d\phi = \infty.$$

Therefore we don't have any information on the value of the integral. So, with this method we can't solve the integral.

(b) $\phi_0 = \pi/2$, $\phi \in [\pi/2, \pi/2 + \pi]$: we close the integration path on the left side.

$$\left|\int_{\Gamma_2} \tilde{Q}(s)\, e^{st} ds\right| \leq \int_{\Gamma_2} \left|\tilde{Q}(s)\right| \left|e^{st}\right| |ds| = \lim_{r\to\infty} \int_{\pi/2}^{3/2\,\pi} \left|\tilde{Q}(s)\right| re^{-rt|\cos\phi|} d\phi$$

Since $\left|\tilde{Q}(re^{i\phi})\right| \to \frac{1}{r}$ for $r \to \infty$, it is sufficient to show that:
$\lim_{r\to\infty} \int_{\pi/2}^{3/2\,\pi} \frac{1}{r}\, re^{-rt|\cos\phi|} d\phi = 0$
in order to prove that $\int_{\Gamma_2} \tilde{Q}(s)\, e^{st} ds = 0$.

$$\int_{\pi/2}^{3/2\,\pi} \frac{1}{r}\, re^{-rt|\cos\phi|} d\phi = \underbrace{\int_{\pi/2}^{\pi/2+\epsilon} e^{-rt|\cos\phi|} d\phi}_{<\epsilon} + \underbrace{\int_{\pi/2+\epsilon}^{3/2\,\pi-\epsilon} e^{-rt|\cos\phi|} d\phi}_{<\pi\cdot e^{-rt\sin\epsilon}} +$$

$$\underbrace{\int_{3/2\,\pi-\epsilon}^{3/2\,\pi} e^{-rt|\cos\phi|} d\phi}_{<\epsilon} < 2\cdot\epsilon + \pi e^{-rt\sin(\epsilon)} \tag{13}$$

It follows that for any small $\delta > 0$ there is r such that:
$\int_{\pi/2}^{3/2,\pi} e^{-rt|\cos\phi|} d\phi < \delta$, therefore, $\lim_{r\to\infty} \int_{\pi/2}^{3/2,\pi} e^{-rt|\cos\phi|} d\phi = 0$

Another way to prove that $\lim_{r\to\infty} \int_{\pi/2}^{3/2\,\pi} \left|\tilde{Q}(s)\right| re^{-rt|\cos\phi|} d\phi = 0$ is to use Bounded Convergence Theorem that allows us to move the $\lim_{r\to\infty}$ into the integrand:

$$\lim_{r\to\infty} \int_{\pi/2}^{3/2\,\pi} \left|\tilde{Q}(s)\right| re^{-rt|\cos\phi|} d\phi = \int_{\pi/2}^{3/2\,\pi} \lim_{r\to\infty} \left|\tilde{Q}(s)\right| re^{-rt|\cos\phi|} d\phi$$

Now, if $\left|\tilde{Q}(re^{i\phi})\right|$ goes to zero with $r \to \infty$ at least as $\frac{1}{r}$ (which is the case here), the integrand goes to zero everywhere, except for $\phi = \pi/2$ and $\phi = 3\pi/2$. Hence, the integral is equal to zero.

From both proofs, we see that the contribution to the integral over the curve Γ_2 is zero, therefore:

$$\int_{p_0-i\infty}^{p_0+i\infty} f(s) ds = 2\pi i \sum_{i=1,2} \mathrm{Res}_i(f(s)). \tag{14}$$

Evaluation of the residues

For equation (10) and equation (14):

$$Q(t) = \frac{1}{2\pi i} \int_{p_0-i\infty}^{p_0+i\infty} \frac{(s+2\delta)Q_0}{(s+\omega_0)(s+\omega_0^*)} e^{st} ds = \sum_{\omega_i=\omega_{1,2}} \lim_{s\to\omega_i} (s-\omega_i) \frac{(s+2\delta)Q_0}{(s+\omega_0)(s+\omega_0^*)} e^{st}$$

where:

$$\omega_1 = -\omega_0, \qquad \omega_2 = -\omega_0^*$$

$$\Rightarrow Q(t) = \lim_{s\to-\omega_0}(s+\omega_0)\frac{(s+2\delta)Q_0e^{st}}{(s+\omega_0)(s+\omega_0^*)} + \lim_{s\to-\omega_0^*}(s+\omega_0^*)\frac{(s+2\delta)Q_0e^{st}}{(s+\omega_0)(s+\omega_0^*)} =$$

$$= \frac{(-\omega_0+2\delta)Q_0\,e^{-\omega_0 t}}{-\omega_0+\omega_0^*} + \frac{(-\omega_0^*+2\delta)Q_0\,e^{-\omega_0^* t}}{-\omega_0^*+\omega_0} \tag{15}$$

and since:

$$-\omega_0 + 2\delta = \delta - i\omega_\gamma, \qquad -\omega_0 + \omega_0^* = -2i\,\omega_\gamma,$$

we can obtain the solution for $t \geq 0$:

$$Q(t) = Q_0\,e^{-\delta t}\left[\cos(\omega_\gamma t) + \frac{\delta}{\omega_\gamma}\sin(\omega_\gamma t)\right]. \tag{16}$$

d) **Verification of the initial conditions and physical interpretation**

$$Q(t=0) = Q_0\,e^{-\delta\cdot 0}\left[\cos(\omega_\gamma\cdot 0) + \frac{\delta}{\omega_\gamma}\sin(\omega_\gamma\cdot 0)\right] = Q_0. \tag{17}$$

$$I(t) = \dot{Q}(t) = Q_0\,e^{-\delta t}\left[\delta\cos(\omega_\gamma t) - \omega_\gamma\sin(\omega_\gamma t) - \delta\cos(\omega_\gamma t) - \frac{\delta^2}{\omega_\gamma}\sin(\omega_\gamma t)\right]$$

$$= -Q_0\frac{\delta^2+\omega_\gamma^2}{\omega_\gamma}\sin(\omega_\gamma t)e^{-\delta t}. \tag{18}$$

$$\Rightarrow I(t=0) = \dot{Q}(t=0) = -Q_0\frac{\delta^2+\omega_\gamma^2}{\omega_\gamma}\sin(\omega_\gamma\cdot 0)e^{-\delta\cdot 0} = 0 \tag{19}$$

Indeed, the solution given by Eq. 16 satisfies the initial conditions.

Equation 16 is describing an oscillatory phenomena with frequency ω_γ damped according to the damping rate δ (figure 5).

The only dissipative term is the resistance R, and clearly $\delta \propto R$. The oscillation frequency is linked to the “active” components L and C.

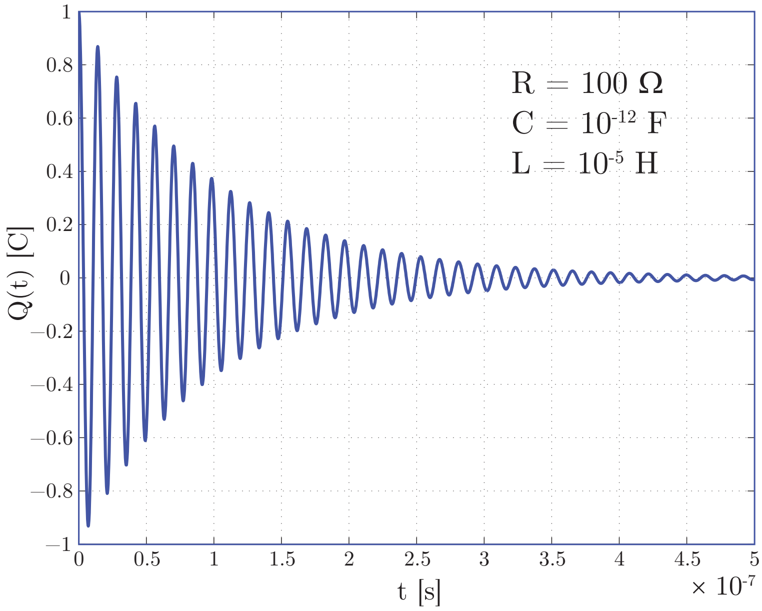

Figure 5: Solution in the real domain.

Exercise 2

In the equation:

$$D(\omega, k) = 1 + \sum_{\alpha} \frac{e^2}{m_\alpha \, \varepsilon_0 k} \int_{-\infty}^{\infty} \frac{dF_{0,\alpha}}{du} \frac{1}{\omega - ku} \, du = 0 \tag{20}$$

we have to consider separately the two species, electrons and ions, with the approximation:

$$k v_{\text{th},i} \ll \omega \ll k v_{\text{th},e}. \tag{21}$$

For the ions, we can use the following expansion:

$$\frac{1}{\omega - ku} \approx \frac{1}{\omega} \left[1 + \frac{ku}{\omega} + \ldots \right] \tag{22}$$

and, for the electrons:

$$\frac{1}{\omega - ku} \approx -\frac{1}{ku}. \tag{23}$$

Now we have to solve the Landau integral for the two species:

Ions

$$\int_{-\infty}^{\infty} \frac{dF_{0,i}}{du} \frac{1}{\omega - ku} \, du \approx \int_{-\infty}^{\infty} \frac{dF_{0,i}}{du} \left[\frac{1}{\omega} + \frac{ku}{\omega^2} \right] du \tag{24}$$

but the integral

$$\frac{1}{\omega} \int_{-\infty}^{\infty} \frac{dF_{0,i}}{du} \, du \tag{25}$$

vanishes because it is the integral over $(-\infty, \infty)$ of an odd function, therefore:

$$\int_{-\infty}^{\infty} \frac{dF_{0,i}}{du} \frac{1}{\omega - ku} \, du \approx \frac{k}{\omega^2} \int_{-\infty}^{\infty} \frac{dF_{0,i}}{du} \, u \, du = \frac{k}{\omega^2} \left[\underbrace{F_{0,i} u \Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} F_{0,i} \, du \right] = -\frac{kn_i}{\omega^2} \tag{26}$$

Electrons

$$\int_{-\infty}^{\infty} \frac{dF_{0,e}}{du} \frac{1}{\omega - ku} \, du \approx -\frac{1}{k} \int_{-\infty}^{\infty} \frac{dF_{0,e}}{du} \frac{du}{u} \tag{27}$$

and, since $F_{0,e}$ is Maxwellian:

$$\frac{dF_{0,e}}{du} = -\frac{u}{v_{\mathrm{th},e}^2} F_{0,e}. \tag{28}$$

We obtain:

$$\int_{-\infty}^{\infty} \frac{dF_{0,e}}{du} \frac{1}{\omega - ku} \, du \approx \frac{1}{kv_{\mathrm{th},e}^2} \int_{-\infty}^{\infty} u \, F_{0,e} \frac{du}{u} = \frac{n_e}{kv_{\mathrm{th},e}^2} \tag{29}$$

Dispersion relation

Combining the results from the previous section:

$$D(\omega, k) \approx 1 - \frac{e^2 \, n_i}{m_i \, \varepsilon_0} \frac{1}{\omega^2} + \frac{e^2 \, n_e}{m_e \, \varepsilon_0} \frac{1}{k^2 v_{\mathrm{th},e}^2} = 1 - \frac{\omega_{\mathrm{p},i}^2}{\omega^2} + \frac{\omega_{\mathrm{p},e}^2}{k^2 v_{\mathrm{th},e}^2} = 0. \tag{30}$$

We want to find an explicit expression for $\omega = \omega(k)$:

$$\omega^2 = \omega_{\mathrm{p},i}^2 \frac{1}{1 + \dfrac{\omega_{\mathrm{p},e}^2}{k^2 v_{\mathrm{th},e}^2}}. \tag{31}$$

The term $\omega_{\mathrm{p},e}^2/(k^2 v_{\mathrm{th},e}^2)$ can be written as:

$$\frac{\omega_{\mathrm{p},e}^2}{k^2 v_{\mathrm{th},e}^2} = \frac{1}{k^2 \lambda_D^2}, \tag{32}$$

where λ_D is the Debye length. In the case of wave lengths $\lambda \sim 1/k$ much bigger than λ_D we have:

$$\frac{\omega_{\mathrm{p},e}^2}{k^2 v_{\mathrm{th},e}^2} \gg 1 \Rightarrow 1 + \frac{\omega_{\mathrm{p},e}^2}{k^2 v_{\mathrm{th},e}^2} \approx \frac{\omega_{\mathrm{p},e}^2}{k^2 v_{\mathrm{th},e}^2} \tag{33}$$

and

$$\omega^2 = \omega_{\mathrm{p},i}^2 \frac{1}{1 + \dfrac{\omega_{\mathrm{p},e}^2}{k^2 v_{\mathrm{th},e}^2}} \approx \frac{\omega_{\mathrm{p},i}^2}{\omega_{p,e}^2} k^2 v_{\mathrm{th},e}^2, \tag{34}$$

but

$$\frac{\omega_{\mathrm{p},i}^2}{\omega_{\mathrm{p},e}^2} v_{\mathrm{th},e}^2 \equiv \frac{m_e}{m_i} \frac{T_e}{m_e} \equiv \frac{T_e}{m_i} = c_s^2, \tag{35}$$

therefore the dispersion relation for the ion-acoustic waves is:

$$\omega^2 = k^2 c_s^2. \tag{36}$$

We still need to verify the initial assumptions. Notice that $\omega/k = c_s$, therefore:

$$\frac{\omega}{k} = c_s \ll v_{\mathrm{th},e} \tag{37}$$

since $c_s/v_{\mathrm{th},e} = \sqrt{m_e/m_i} \ll 1$, and

$$\frac{\omega}{k} = c_s \gg v_{\mathrm{th},i}, \tag{38}$$

that is satisfied if $c_s/v_{\mathrm{th},i} = \sqrt{T_e/T_i} \gg 1$.

Notice that the condition $T_e/T_i \gg 1$ is necessary to avoid a strong damping of the ion-acoustic wave.