Plasma Physics I

Solution to the Series 6 (October 18, 2025)

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Exercise 1

a Let us **consider a cold plasma**: $T = 0 \rightarrow p = 0$. The momentum equation for the plasma species α (keeping the collision term) is:

$$m_\alpha \frac{d\mathbf{v}_\alpha}{dt} = q_\alpha(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) - \nu_\alpha m_\alpha \mathbf{v}_\alpha \tag{1}$$

The collision frequency ν_α for species α is taken as a generic coefficient, which could be due to collisions with neutrals or with charged particles. Since we want to determine the main effects of collisions on longitudinal waves, we model collisions in the simplest possible way, namely assuming ν_α to be a constant.

Assuming a uniform equilibrium with $\mathbf{B}_0 = 0$, $\mathbf{E}_0 = 0$ and $\mathbf{v}_{\alpha,0} = 0$, the linearised momentum equation (using the notation $\mathbf{v}_{\alpha,1} = \mathbf{v}_\alpha$, $\mathbf{E}_1 = \mathbf{E}$) is:

$$m_\alpha \frac{\partial \mathbf{v}_\alpha}{\partial t} = q_\alpha \mathbf{E} - \nu_\alpha m_\alpha \mathbf{v}_\alpha. \tag{2}$$

The Fourier tranformation gives:

$$-i\omega \mathbf{v}_\alpha = \frac{q_\alpha}{m_\alpha}\mathbf{E} - \nu_\alpha \mathbf{v}_\alpha \tag{3}$$

and, for the velocity $\mathbf{v}_\alpha$:

$$\mathbf{v}_\alpha = \frac{q_\alpha/m_\alpha}{\nu_\alpha - i\omega}\mathbf{E} = i\frac{q_\alpha/m_\alpha}{\omega + i\nu_\alpha}\mathbf{E} \tag{4}$$

Now we have a relation between the velocity and the electric field. We can then find the relation $\mathbf{j} = \mathbf{j}(\mathbf{E})$ (constitutive equation):

$$\mathbf{j} = \sum_\alpha q_\alpha n_{0\alpha} \mathbf{v}_\alpha = i\epsilon_0 \sum_\alpha \frac{q_\alpha^2 n_{0\alpha}}{\epsilon_0 m_\alpha(\omega + i\nu_\alpha)}\mathbf{E} \tag{5}$$

therefore the conductivity is:

$$\sigma = i\epsilon_0 \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega + i\nu_\alpha} \tag{6}$$

We can notice that the system is isotropic, since nothing is perturbing the symmetry (for example a magnetic field $\mathbf{B}_0 \neq 0$). Therefore, the conductivity is a scalar quantity.

The dielectric tensor can be evaluate with the following expression:

$$\epsilon = \mathbf{1}\left(1 + \frac{i}{\epsilon_0 \omega}\,\sigma\right) = \mathbf{1}\left(1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega(\omega + i\nu_\alpha)}\right) = \epsilon\,\mathbf{1} \tag{7}$$

The wave equation for $\mathbf{E}$ is:

$$\underbrace{\left[N^2\left(\frac{\mathbf{kk}}{k^2} - \mathbf{1}\right) + \vec{\epsilon}\right]}_{\vec{D}} \cdot \mathbf{E} = 0 \;\rightarrow\; \vec{D}\cdot\mathbf{E} = 0 \tag{8}$$

and, choosing $\mathbf{k} = k\hat{\mathbf{e}}_\mathbf{z}$

$$\mathbf{kk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k^2 \end{pmatrix} \Rightarrow \vec{D} = N^2\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} = \tag{9}$$

$$= \begin{pmatrix} \epsilon - N^2 & 0 & 0 \\ 0 & \epsilon - N^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \tag{10}$$

We find the following dispersion relation:

$$\det \vec{D} = 0 \;\Rightarrow\; \epsilon\left(\epsilon - N^2\right)^2 = 0 \tag{11}$$

which has one solution for the longitudinal wave (Langumuir wave) and two coincident solutions for the tranverse waves.

b The dispersion relation for the Langmuir wave that we want to study is:

$$\epsilon = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega(\omega + i\nu_\alpha)} = 0 \tag{12}$$

Using the condition $\omega_{\mathrm{pe}} \gg \omega_{\mathrm{p}i}$ we can simplify the dispersion relation:

$$\omega_{\mathrm{pe}}^2 \approx \omega(\omega + i\nu_e) \Rightarrow \omega^2 + i\omega\nu_e - \omega_{\mathrm{pe}}^2 \approx 0 \tag{13}$$

where the frequency ω has an imaginary part:

$$\omega = \omega_R + i\omega_I \tag{14}$$

Using the eq. (14) in the eq. (13) we find:

$$\omega_R^2 - \omega_I^2 = \omega_{\mathrm{pe}}^2 + \nu_e\omega_I, \qquad \text{(real part)} \tag{15}$$

$$2i\omega_R\omega_I = -i\nu_e\omega_R, \qquad \text{(imaginary part)} \tag{16}$$

and finally:

$$\omega_R = \sqrt{\omega_{\mathrm{pe}}^2 - \frac{\nu_e^2}{4}} \tag{17}$$

$$\omega_I = -\frac{\nu_e}{2} \tag{18}$$

Therefore, if we consider an electric field with magnitude $E \propto e^{-i\omega t}$, we have:

$$E \propto e^{-i\omega t} = e^{-i\omega_R t}\, e^{-\frac{\nu_e}{2}t} \tag{19}$$

The second exponential in the equation shows the effect of the damping due to the collisions.

Exercise 2

a The neutron star emits electromagnetic radiation, which travels through interstellar plasma and therefore is submited to dispersion. There may also be longitudinal waves induced in such plasma, but these can not propagate in vacuum. Therefore the waves that will arrive to the antenna on Earth are only electromagnetic transverse waves.

The dispersion relation for a tranverse wave in unmagnetised plasmas is:

$$\omega = \sqrt{\omega_{\mathrm{pe}}^2 + k^2c^2} \tag{20}$$

which can be derived from the linearised cold fluid equations in the case of no magnetic field, or taking the limit $B \to 0$ in the general dispersion relation for a cold magnetised plasma.

For example, in the lecture we have seen that for transverse waves with $\theta = 0$ (propagation along the equilibrium $\mathbf{B}$), the dispersion relation is

$$N^2 = \frac{k^2c^2}{\omega^2} = \frac{(\omega + \omega_L)(\omega - \omega_R)}{(\omega - \Omega_{ci})(\omega - \Omega_{ce})} \tag{21}$$

If we take $B \to 0$, we have $\omega_L, \omega_R \to \omega_{pe}$ and $\Omega_{ci}, \Omega_{ce} \to 0$, thus recoverig the previous dispersion relation.

The group velocity (velocity of the information coming from the pulsar and reaching the earth):

$$v_g(\omega) \equiv \frac{\partial \omega}{\partial k} = \frac{kc^2}{\sqrt{\omega_{\mathrm{pe}}^2 + k^2c^2}} = \frac{\sqrt{\omega^2 - \omega_{\mathrm{pe}}^2}}{\omega}\, c \tag{22}$$

where $k = \frac{1}{c}\sqrt{\omega^2 - \omega_{\mathrm{pe}}^2}$.

The information from the pulsar is reaching the earth after:

$$t = \frac{x}{v_g} = \frac{x}{c}\frac{\omega}{\sqrt{\omega^2 - \omega_{\mathrm{pe}}^2}} \simeq \frac{x}{c}\left(1 + \frac{1}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right) \tag{23}$$

supposing $\omega \gg \omega_{\mathrm{pe}}$.
The total differential of the eq.(23) is:

$$\mathrm{d}t = \frac{\partial}{\partial \omega}\left(\frac{x}{c}\left(1 + \frac{1}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)\right)\mathrm{d}\omega = -\frac{x}{c}\frac{\omega_{\mathrm{pe}}^2}{\omega^3}\,\mathrm{d}\omega \tag{24}$$

therefore:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{c}{x}\frac{\omega^3}{\omega_{\mathrm{pe}}^2} \qquad \Rightarrow \qquad \frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{c}{x}\frac{f^3}{f_p^2} \tag{25}$$

since $f = \omega/2\pi$.

b From the equation (25), we can evaluate the distace of the pulsar: $x = -\frac{c}{\mathrm{d}f/\mathrm{d}t}\frac{f^3}{f_p^2}$. Having $n_e = 2 \times 10^6$ m^{-3}, $f = 80$ MHz and $df/dt = -5$ MHz/s, we find

$f_p[\mathrm{Hz}] = 8.98 \times 10^3\sqrt{n_e[\mathrm{cm}^{-3}]} \simeq 9\sqrt{n_e[\mathrm{m}^{-3}]} = 1.27 \times 10^4$ Hz and

$x = 1.91 \times 10^{17}$ m $= 6.6$ parsec.