Plasma Physics I

Solution to the Series 3 (September 27, 2025)

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Exercise 1

For the collisions with small deflection angle, the momentum transfer cross-section is:

$$\sigma_p = \sigma_{E_k} \frac{m_1 + m_2}{2m_1} = \frac{q_1^2 q_2^2}{2\pi\epsilon_0^2} \frac{\ln\Lambda}{m_1 m_2 v^4} \frac{m_1 + m_2}{2m_1} \tag{1}$$

with $b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0 \mu v^2}$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ we can write this as follows

$$\sigma_p = 4\pi b_{90}^2 \ln\Lambda \frac{m_2}{m_1 + m_2} \tag{2}$$

The cross-section σ_p has to be compared with the cross-section for the collisions with a deflection angle (see remark at end) $\theta \geq 90°$:

$$\sigma_{90} = \pi b_{90}^2 \tag{3}$$

The ratio between them is:

$$\frac{\sigma_p}{\sigma_{90}} = 4\ln\Lambda \frac{m_2}{m_1 + m_2} \tag{4}$$

and for $m_2 \gg m_1$ (i.e., for electrons colliding with nucleus of deuterium or tritium):

$$\frac{\sigma_p}{\sigma_{90}} \approx 4\ln\Lambda \tag{5}$$

If $T_e = T_i = 1$ keV the coulomb logarithm is (formulary NRL, page 34)

$$\ln\Lambda = 24 - \ln\left(\frac{\sqrt{n_e[cm^{-3}]}}{T[eV]}\right) \approx 24 - \ln\left(\frac{\sqrt{10^{14}}}{10^3}\right) \approx 15 \tag{6}$$

Therefore $\frac{\sigma_p}{\sigma_{90}} \approx 60 \gg 1$. In general, for $m_2 \gg m_1$ with a typical temperature of a plasma used in thermonuclear fusion applications, we have $\frac{\sigma_p}{\sigma_{90}} \gg 1$. We can then conclude that it's possible to neglect the effects of the collisions with $\theta \geq 90°$.

Remark: In principle, we should compare σ_p to $\sigma_{90,p}$, the momentum transfer cross-section for deflection angles $\theta \geq 90°$, which is not the same as σ_{90}. However, we can show that the difference is small:

$$\sigma_{90,p} = \frac{\nu_{90,p}}{nv} = \frac{1}{nv}\frac{1}{|\vec{p}|}\frac{d|\vec{p}|}{dt}\bigg|_{\theta \geq 90} \tag{7}$$

$$= \frac{1}{nv}\frac{1}{mv}\int_0^{b_{90}} \Delta p n v d\sigma \tag{8}$$

$$= \frac{m_2}{m_1+m_2}\frac{p}{mv}\int_0^{b_{90}} (1-cos\theta) 2\pi b db \tag{9}$$

$$= \frac{p}{mv}\int_0^{b_{90}} (1-cos\theta) 2\pi b db \tag{10}$$

$$= \int_0^{b_{90}} (1-cos\theta) 2\pi b db \tag{11}$$

where we used $\Delta p = p\frac{m_2}{m_1+m_2}(1-cos\theta)$ (see Fasoli, App. B), $d\sigma = 2\pi b db$ and the final equality holds in the limit $\frac{m_2}{m_1+m_2} \approx 1$ as $m_2 \gg m_1$.

In the integral, $(1-cos\theta)$ varies between 1 at $\theta = 90$ and 2 at $\theta = 180°$. Thus,

$$\sigma_{90} = \int_0^{b_{90}} 1 \times 2\pi b db < \sigma_{90,p} < \int_0^{b_{90}} 2 \times 2\pi b db = 2\sigma_{90} \tag{12}$$

Exercise 2

Consider the collisions between electrons and ions. Having $v_i \ll v_e$ and $m_2 \gg m_1$, it's possible to neglect the velocity of the ions ($v_i \approx 0$): the results for σ_p that we have found for the centre-of-mass reference frame are then also valid in the laboratory reference frame.
For the electrons with velocity v_e, the momentum transfer rate due to the collisions with ions (ion density n_i) is:

$$\left|\frac{dp}{dt}\right| = \nu_p^{e/i} \cdot p \tag{13}$$

where

$$\nu_p^{e/i} = n_i \sigma_p v_e \tag{14}$$

with σ_p given by $\sigma_p = \frac{q_1^2 q_2^2}{2\pi\epsilon_0^2} \frac{\ln\Lambda}{m_1 m_2 v^4} \frac{m_1+m_2}{2m_1}$ (see exercise 1). With $m_1 = m_e \ll m_2 = M_i$, $q_1 = -e$ and $q_2 = Ze$, we find:

$$\nu_p^{e/i} = n_i \frac{Z^2 e^4}{4\pi\varepsilon_0^2} \frac{\ln\Lambda}{m_e^2 v_e^3} \tag{15}$$

The *total* momentum lost per second (in the 3 spatial directions) is given by $\dfrac{d\vec{p}}{dt} = -\nu_p^{e/i} m_e \vec{v}$. Now we have to average this quantity:

$$\left\langle \frac{d\vec{p}}{dt} \right\rangle = -\int \nu_p^{e/i}\, f_e(\vec{v}) m_e \vec{v}\, d^3v \tag{16}$$

The normalized distribution function is:

$$f_e(\vec{v}) = \left(\frac{m_e}{2\pi\, T_e}\right)^{3/2} \exp\left[-\frac{m_e(\vec{v}-\vec{v}_d)^2}{2\, T_e}\right] \tag{17}$$

where $\vec{v}_d = v_d \hat{x}$ is the drift velocity in the x direction. Since $v_d \ll v_{\text{the}} = (T_e/m_e)^{1/2}$

$$\frac{m_e(\vec{v}-\vec{v}_d)^2}{2\, T_e} = \frac{v^2}{2\, v_{\text{the}}^2} + \frac{v_d^2}{2\, v_{\text{the}}^2} - \frac{\vec{v}\cdot\vec{v}_d}{v_{\text{the}}^2} \approx \frac{v^2}{2\, v_{\text{the}}^2} - \frac{\vec{v}\cdot\vec{v}_d}{v_{\text{the}}^2} = \frac{v^2}{2\, v_{\text{the}}^2} - \frac{v_x\, v_d}{v_{\text{the}}^2} \tag{18}$$

we can approximate the distribution function:

$$\begin{aligned} f_e(\vec{v}) &= \left(\frac{m_e}{2\pi\, T_e}\right)^{3/2} \exp\left[-\frac{v^2}{2\, v_{\text{the}}^2} + \frac{v_x\, v_d}{v_{\text{the}}^2}\right] \\ &= \left(\frac{m_e}{2\pi\, T_e}\right)^{3/2} \exp\left[-\frac{v^2}{2\, v_{\text{the}}^2}\right] \exp\left[\frac{v_x\, v_d}{v_{\text{the}}^2}\right] \qquad (19) \\ &\approx \left(\frac{m_e}{2\pi\, T_e}\right)^{3/2} \exp\left[-\frac{v^2}{2\, v_{\text{the}}^2}\right] \left[1 + \frac{v_x\, v_d}{v_{\text{the}}^2}\right] \\ &= f_{e0}(\vec{v}) \left[1 + \frac{v_x\, v_d}{v_{\text{the}}^2}\right] \qquad (20) \end{aligned}$$

where $f_{e0}(\vec{v})$ is the maxwellian distribution function with $v_d = 0$:

$$f_{e0}(\vec{v}) = \left(\frac{m_e}{2\pi\, T_e}\right)^{3/2} \exp\left[-\frac{v^2}{2\, v_{\text{the}}^2}\right] \tag{21}$$

Therefore, the component in the x direction of $\langle d\vec{p}/dt\rangle$ is:

$$\begin{aligned}\left\langle \frac{dp_x}{dt} \right\rangle &= -\int_{-\infty}^{+\infty} \left[1 + \frac{v_x\, v_d}{v_{\text{the}}^2}\right] f_{e0}(\vec{v})\, \nu_p^{e/i}\, m_e v_x\, d^3\vec{v} \\ &= -\int_{-\infty}^{+\infty} f_{e0}(\vec{v})\, \nu_p^{e/i}\, m_e v_x\, d^3\vec{v} \\ &\quad -\int_{-\infty}^{+\infty} \frac{v_x v_d}{v_{\text{the}}^2} f_{e0}(\vec{v})\, \nu_p^{e/i} m_e v_x\, d^3\vec{v} \end{aligned} \tag{22}$$

The first term in the right-side of the equation (22) is zero, because it's the integral of an anti-symmetric function, f_{e0} and $\nu_p^{e/i}$ are symmetric[1], $v_x \in (-\infty, \infty)$ is anti-symmetric.

From the *physical* point of view, the total momentum of the unperturbed electron population (ie $v_d = 0$) is zero and the total momentum of the targets is also zero due to the approximation $v_e \gg v_i \approx 0$. Therefore, it's not possible to have a net transfer of momentum $\vec{p}$ between the two populations. This is of course also true in the others directions (y and z).

Solving the integral:

$$\begin{aligned}\left\langle \frac{dp_x}{dt} \right\rangle &= -\int \frac{v_x\, v_d}{v_{\text{the}}^2}\, f_{e0}(\vec{v})\, \nu_p^{e/i}\, m_e v_x\, d^3\vec{v} = \\ &= -n_i \frac{Z^2 e^4}{4\pi\varepsilon_0^2} \frac{\ln\Lambda}{m_e v_{\text{the}}^2} v_d \underbrace{\int f_{e0}(\vec{v})\, \frac{v_x^2}{v^3}\, d^3\vec{v}}_{I_1}\end{aligned} \tag{23}$$

where we used the approximation $\ln\Lambda$ independent of v.

In general, we have $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$. Since the function $f_{e0}(\vec{v})$ is symmetric in the space of the variables (v_x, v_y, v_z), we can reduce the problem to a single dimension: $v_x^2 = \frac{1}{3}v^2$. Using a spherical geometry, we have: $d^3v = 4\pi v^2 dv$.

And then we find:

$$\begin{aligned} I_1 &= \int f_{e0}(\vec{v})\, \frac{v_x^2}{v^3}\, d^3\vec{v} = \frac{1}{3}\int f_{e0}(v)\, \frac{v^2}{v^3}\, 4\pi v^2 dv \\ &= \frac{1}{3}\int_0^{\infty} f_{e0}(v)\, 2\pi(2v\, dv) = \frac{2\pi}{3}\int_0^{\infty} f_{e0}(v)\, dv^2 \\ &= \frac{2\pi}{3} \frac{1}{(2\pi)^{3/2} v_{\text{the}}^3} \underbrace{\int_0^{\infty} \exp\left[-\frac{v^2}{2\, v_{\text{the}}^2}\right] dv^2}_{I_2} \end{aligned} \tag{24}$$

To solve the integral I_2 we can make the substitution:

$$\xi = v^2/(2\, v_{\text{the}}^2) \;\Rightarrow\; dv^2 = 2\, v_{\text{the}}^2\, d\xi \tag{25}$$

[1] $\nu_p^{e/i}$ is in inverse proportion with the cube of v but v is a quadratic function of the components of the vector $\underline{v}$: $\nu_p^{e/i} \propto v^{-3} = \left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right)^{-3}$

therefore:

$$I_2 = \int_0^\infty \exp\left[-\frac{v^2}{2\,v_{\rm the}^2}\right] dv^2 = 2\,v_{\rm the}^2 \int_0^\infty e^{-\xi} d\xi = \\ = 2\,v_{\rm the}^2 \left[-e^{-\xi}\right]_0^\infty \equiv 2\,v_{\rm the}^2 \tag{26}$$

and

$$I_1 = \frac{2\pi}{3}\frac{1}{(2\pi)^{3/2} v_{\rm the}^3}\, 2\,v_{\rm the}^2 = \frac{4\pi}{3(2\pi)^{3/2}}\frac{1}{v_{\rm the}} = \frac{1}{3}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{v_{\rm the}} \tag{27}$$

The momentum of the electron population is $p = m_e v_d$. The *average* of the momentum loss rate is:

$$\begin{aligned} \overline{\nu}_p^{e/i} &= \frac{1}{p}\left|\left\langle \frac{dp_x}{dt}\right\rangle\right| = \frac{1}{m_e v_d} n_i \frac{Z^2 e^4}{4\pi\varepsilon_0^2}\frac{\ln\Lambda}{m_e v_{\rm the}^2}\, v_d\, \frac{1}{3}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{v_{\rm the}} \\ &= \underbrace{\frac{n_i Z^2 e^4}{4\pi\varepsilon_0^2}\frac{\ln\Lambda}{m_e^2 v_{\rm the}^3}}_{\nu_p^{e/i}(v_{\rm the})} \left[\frac{1}{3}\left(\frac{2}{\pi}\right)^{1/2}\right] \end{aligned} \tag{28}$$

and finally we can conclude that:

$$\overline{\nu}_p^{e/i} = \left[\frac{1}{3}\left(\frac{2}{\pi}\right)^{1/2}\right] \nu_p^{e/i}(v_{\rm the}) \approx 0.26 \cdot \nu_p^{e/i}(v_{\rm the}) \tag{29}$$

We could also suppose $\overline{\nu}_p^{e/i} \sim \nu_p^{e/i}(v_{\rm the})$, overestimating $\overline{\nu}_p^{e/i}$.

Exercise 3

The α particles will collide with the three species (electrons, deuterium ions and tritium ions) losing their energy and therefore their velocity. The estimated relaxation time is necessarily a function of the α's velocity.

The characteristic velocity of the particles is the thermal velocity. Using the given parameters, we obtain:

$$v_{\text{th},e} = \sqrt{\frac{T_e}{m_e}} = 4.19 \times 10^7 \text{ m/s}$$
$$v_{\text{th},D} = \sqrt{\frac{T_D}{2\,m_p}} = 6.92 \times 10^5 \text{ m/s}$$
$$v_{\text{th},T} = \sqrt{\frac{T_T}{3\,m_p}} = 5.65 \times 10^5 \text{ m/s}$$

The α's velocity is given by their energy:

$$v_\alpha = \sqrt{\frac{2\,E_\alpha}{4\,m_p}} = 6.92 \times 10^3 \sqrt{E_\alpha[eV]} \text{ m/s}$$

For $E_\alpha = 3.5$ MeV, $v_\alpha = 1.3 \times 10^7$ m/s and therefore:

$$v_{\text{th},T} \lesssim v_{\text{th},D} \ll v_\alpha < v_{\text{th},e} \tag{30}$$

The last relation is valid as long as $E_\alpha \gg 10$ keV. For E_α down to 100 keV for example, well below the initial 3.5 MeV, we still have $v_\alpha \approx 3v_{\text{th},D}$. (30) is therefore the regime important to determine the main energy loss of the α's. What happens for $E_\alpha \lesssim 10 - 100$ keV is therefore neglected here.

The general form of ν_{E_k} for collisions of particles of the species j (projectiles) upon particles of the species k (targets) is

$$\nu_{E_k}^{j/k} \sim n_k \frac{Z_j^2\, Z_k^2 e^4}{2\pi\varepsilon_0^2} \frac{\ln \Lambda_k}{m_j\, m_k v_j^3}$$

Therefore the three collision frequencies taking part in the energy transfer process are:

$$\bar{\nu}_{E_k}^{\alpha/e} \simeq \frac{1}{3}\sqrt{\frac{2}{\pi}}\, n_e \frac{4e^4}{2\pi\varepsilon_0^2} \frac{\ln \Lambda}{4\,m_p m_e} \frac{1}{v_{the}^3} \approx 5.5 \text{ s}^{-1} \tag{31}$$
$$\nu_{E_k}^{\alpha/D} \simeq \frac{n_e}{2} \frac{4e^4}{2\pi\varepsilon_0^2} \frac{\ln \Lambda}{4\,m_p\, 2\,m_p} \frac{1}{v_\alpha^3} = \nu_{E_k}^{\alpha/e} \sqrt{\frac{9\pi m_p}{4\,m_e}} \left(\frac{T}{E_\alpha}\right)^{3/2} \tag{32}$$
$$\nu_{E_k}^{\alpha/T} = \frac{2}{3}\, \nu_E^{\alpha/D} \tag{33}$$

where we have used the relations:

$$\ln \Lambda = 24 - \ln\left(\frac{\sqrt{n_e[\text{cm}^{-3}]}}{T[\text{eV}]}\right) \approx 24 - \ln\left(\frac{\sqrt{10^{14}}}{10^4}\right) \approx 17.1$$

and

$$n_D = n_T = 0.5 n_e$$

Strictly speaking, the velocity in the denominator is relative to the centre-of-mass reference frame. Usually, this relative velocity is dominated by the velocity of one particle (target particle or on-coming particle). That is the case for the three interactions studied and that's why the electron thermal velocity (target particles) is present in the expression of $\bar{\nu}_{E_k}^{\alpha/e}$ ($v_{\mathrm{th},e} \gg v_\alpha$). The factor $\frac{1}{3}\sqrt{\frac{2}{\pi}}$ in the expression of $\bar{\nu}_{E_k}^{\alpha/e}$ is due to the Maxwellian distribution of the electron velocities. It can be obtained from an integration over the electron distribution function or from the general expressions in the NRL Plasma Formulary on page 31.

At the beginning of the relaxation process ($E_\alpha = 3.5$ MeV), the frequencies are equal to:

$$\nu_{E_k}^{\alpha/D} \approx 9.4 \times 10^{-2}\ \mathrm{s}^{-1} \qquad \nu_{E_k}^{\alpha/T} \approx 6.3 \times 10^{-2}\ \mathrm{s}^{-1}$$

therefore with $\sqrt{\frac{9\pi m_p}{4 m_e}} \left(\frac{T}{E_\alpha}\right)^{3/2} \approx 1.7 \times 10^{-3}$, we have $\nu_{E_k}^{\alpha/D}, \nu_{E_k}^{\alpha/T} \ll \bar{\nu}_{E_k}^{\alpha/e} \approx 5.5\ \mathrm{s}^{-1}$: the α particles transmit their energy to the electrons.

In this case, the relaxation time is given by:

$$\tau = \left(\nu_{E_k}^{\alpha/D} + \nu_{E_k}^{\alpha/T} + \bar{\nu}_{E_k}^{\alpha/e}\right)^{-1} \approx \left(\bar{\nu}_{E_k}^{\alpha/e}\right)^{-1} = 183\ \mathrm{ms}$$

We can conclude that:

- At the beginning the electrons are more effective in the α thermalisation process. Only when $E_\alpha \lesssim \left(\frac{9\pi m_p}{4 m_e}\right)^{1/3} T$, or $E_\alpha \lesssim 240$ keV, the deuterium ions are more effective than the electrons (the tritium ions are playing an important role in the thermalisation process for lower energies).

- Since the energy transfer process due to the collisions between α particles and electrons during the transition $E_\alpha = 3.5 MeV \to 240$ keV is more important than the transition $E_\alpha = 240$ keV $\to 10$ keV, the electrons are heated more than the ions.