Plasma I

Solution to the Series 1 (September 13, 2025)

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Exercise 1

The definition of a maxellian distribution of a population of particles α with velocity $\vec{v}$ is:

$$f_\alpha(\vec{v}) = \left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2} \exp\left(-\frac{m_\alpha}{2\,T_\alpha}\vec{v}^2\right)$$

Note that the distribution function depends on the vector $\vec{v}$; therefore, in the evaluation of the average, it's necessary to include the direction of the velocity $\vec{v}$. Using the definition of the thermal velocity $v_{th,\alpha} = \sqrt{\frac{T_\alpha}{m_\alpha}}$, we can derive the following expression for the maxwellian distribution f_α as a function of $v_{th,\alpha}$:

$$f_\alpha = \frac{1}{(2\pi)^{3/2}} \frac{1}{v_{th,\alpha}^3} \exp\left(-\frac{\vec{v}^2}{2\,v_{th,\alpha}^2}\right)$$

Definition of the n^{th} moment of a variable x for distribution function $f(x)$:

$$\langle x^n \rangle = \int_{-\infty}^{+\infty} x^n f(x) dx$$

Using this definition, it's possible to write the integral for the average value of v_x^2:

$$\left\langle v_x^2 \right\rangle = \frac{1}{(2\pi)^{3/2}} \frac{1}{v_{th,\alpha}^3} \int_{-\infty}^{+\infty} v_x^2\, e^{-\frac{v_x^2+v_y^2+v_z^2}{2\,v_{th,\alpha}^2}}\, dv_x\, dv_y\, dv_z =$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{v_{th,\alpha}^3} \underbrace{\int_{-\infty}^{+\infty} v_x^2\, e^{-\frac{v_x^2}{2\,v_{th,\alpha}^2}} dv_x}_{\sqrt{2\pi}\, v_{th,\alpha}^3} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{v_y^2}{2\,v_{th,\alpha}^2}} dv_y}_{\sqrt{2\pi}\, v_{th,\alpha}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{v_z^2}{2\,v_{th,\alpha}^2}} dv_z}_{\sqrt{2\pi}\, v_{th,\alpha}}$$

where the following solution for the integrals has been used (see Appendix A for the cases n=0 and n=1):

$$I(a) = \int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0 \tag{1}$$

$$I_n(a) = \int_{-\infty}^{+\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \sqrt{\frac{\pi}{a^{2n+1}}} \qquad a > 0,\, n \in N \cup \{0\} \tag{2}$$

Finally, we can write the *root–mean–square* (RMS) velocity:

$$\langle v_x^2 \rangle = v_{th,\alpha}^2 \qquad \Rightarrow \qquad \sqrt{\langle v_x^2 \rangle} = v_{th,\alpha} \tag{3}$$

We can use this result to estimate $\langle \vec{v}^2 \rangle$. Since:

$$\vec{v}^2 = v_x^2 + v_y^2 + v_z^2$$

and v_x, v_y, v_z are independent variables, we have:

$$\langle v^2 \rangle = \langle v_x^2 + v_y^2 + v_z^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle = 3 \langle v_x^2 \rangle = 3\, v_{th,\alpha}^2$$

and thus:

$$\sqrt{\langle v^2 \rangle} = \sqrt{3}\, v_{th,\alpha}$$

To estimate the averaged value of $|\vec{v}|$ (first momentum of the velocity), it's better to solve the integral using spherical coordinates. For the integration of the volume element of the velocity d^3v we have:

$$\int_{-\infty}^{+\infty} d^3v \qquad \Rightarrow \qquad \underbrace{\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi} \int_0^{+\infty} v^2 dv$$

so we can write:

$$\langle |\vec{v}| \rangle = 4\pi \int_0^{+\infty} v f_\alpha v^2 dv = 4\pi \frac{1}{(2\pi)^{3/2}} \frac{1}{v_{th,\alpha}^3} \int_0^{+\infty} v\, e^{-\frac{v^2}{2 v_{th,\alpha}^2}} v^2 dv \tag{4}$$

Using the new variable defined as $x = \frac{v^2}{2 v_{th,\alpha}^2}$, we have:

$$x = \frac{v^2}{2\, v_{th,\alpha}^2} \qquad \Rightarrow \qquad v^2 = 2\, v_{th,\alpha}^2\, x$$

$$dx = \frac{v}{v_{th,\alpha}^2} dv \qquad \Rightarrow \qquad v\, dv = v_{th,\alpha}^2 dx$$

The integral (4) can be rewritten as:

$$\langle |\vec{v}| \rangle = 4\pi \frac{1}{(2\pi)^{3/2}} \frac{1}{v_{th,\alpha}^3} 2\, v_{th,\alpha}^4 \int_0^{+\infty} x\, e^{-x} dx = \frac{4}{\sqrt{2\pi}} v_{th,\alpha} \int_0^{+\infty} x\, e^{-x} dx$$

solving it by parts:

$$\int_0^{+\infty} x\, e^{-x} dx = -x\, e^{-x}\Big|_0^{\infty} + \int_0^{+\infty} e^{-x} dx = 0 - e^{-x}\Big|_0^{\infty} = 1$$

and finally,

$$\langle |\vec{v}| \rangle = \sqrt{\frac{8}{\pi}}\, v_{th,\alpha}$$

Exercise 2

a) The number of electrons injected in the tube per unit time is:

$$F_0 = \frac{\text{electric current}}{\text{electron charge}} = \frac{1.0 \text{ A}}{1.6 \cdot 10^{-19} \text{ C}} = 6.24 \cdot 10^{18} \text{s}^{-1}$$

The quantity F_0 corresponds to a flux $n_e v_e S$, where S is the section of the electron beam.

Collisions between electrons and neutrals are inelastic. The energy lost by a colliding electron is such that after the collision it has insufficient energy to ionize further neutrals. Hence, the total rate of ionization in the tube is equal to the number of electrons lost from the beam between its entrance and exit of the tube. We therefore need to calculate the decay of the electron flux $F(x)$ along the tube.

In the steady state assumed here, the electron flux $F(x+dx)$ at the position $x + dx$ along the beam differs from the electron flux $F(x)$ at position x due to the fact that a fraction f_p of the electrons are lost due to ionization between x and $x + dx$:

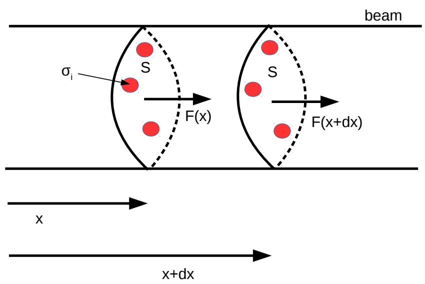
$$F(x + dx) = F(x) - f_p F(x) \tag{5}$$

where

$$f_p = \frac{\text{(target density) x (volume) x (cross section of a single target)}}{\text{beam area}} = \frac{n_n S dx \sigma_{\text{ion}}}{S} \tag{6}$$

which we note is independent of S. We thus find that

$$dF = F(x + dx) - F(x) = -f_p F(x) = -n_n \sigma_{\text{ion}} dx F \tag{7}$$



$$\frac{dF}{dx} = -n_n \sigma_{ion} F = -\frac{F}{\lambda_{\text{mfp}}} \qquad \Rightarrow \qquad \frac{dF}{F} = -\frac{dx}{\lambda_{\text{mfp}}}$$

the solution is:

$$F(x) = F_0 \exp\left(-\frac{x}{\lambda_{\text{mfp}}}\right)$$

The quantity $\lambda_{\text{mfp}} = (n_n \sigma_{ion})^{-1}$ is the <u>m</u>*ean* <u>f</u>*ree* <u>p</u>*ath*. The total number of ions per second produced in the tube is given by the difference between the incoming and outgoing electron flux:

$$\frac{dN}{dt} = F_{in} - F_{out} = F_0 - F(x = l) = F_0 \left[1 - \exp\left(-\frac{l}{\lambda_{\text{mfp}}}\right)\right]$$

To have a numerical solution it's necessary to estimate the neutral particle density, n_n. We have that $n_n = \frac{N_n}{V} = \frac{n\mathcal{A}}{V}$, where N_n is the total number of atoms equal to the number of moles n times the Avogadro's number $\mathcal{A}$. To estimate n, it's possible to use the ideal gas law $pV = nRT \rightarrow \quad n = \frac{pV}{RT}$.

The following expression can then be derived:

$$n_n = \frac{P_{[\text{Pa}]}\mathcal{A}_{[\text{mol}^{-1}]}}{R_{[\text{J}\cdot\text{K}^{-1}\text{mol}^{-1}]}T_{[\text{K}]}}$$

Given that 1 Pa $= 7.501 \times 10^{-3}$ Torr, then:

$$n_n = \frac{10^{-3} \cdot 6.02 \times 10^{23}}{8.31 \cdot 300 \cdot 7.501 \times 10^{-3}} \simeq 3.3 \times 10^{19} \text{ m}^{-3}.$$

It's possible now to derive the mean free path:

$$\lambda_{\text{mfp}} = \frac{1}{n_n \sigma_{ion}} = \frac{1}{3.3 \times 10^{19} \text{ m}^{-3} \cdot 10^{-20} \text{ m}^2} \approx 3 \text{ m}$$

$$\frac{dN}{dt} = 6.24 \times 10^{18} \left[1 - \exp\left(-\frac{2}{3}\right)\right] = 3 \times 10^{18} \text{ ions} \cdot \text{s}^{-1}$$

Alternative solution using collision frequency:

The ionization frequency (per electron) is given by $\nu_{ion} = n_n \sigma_{ion} v_e$. This means that the ionization source (=ionization per second and per m^3) is, locally, given by $S_{ion} = \nu_{ion} \cdot n_e = n_n \sigma_{ion} v_e n_e$

Now, we need to integrate this over the entire beam volume, taking into account that the density of beam electrons, $n_e(x)$, is decreasing along the way:

$$\text{Ionizations/sec} = \frac{dN}{dt} = \int_{V_{beam}} n_n \sigma_{ion} v_e n_e(x) dV$$

Let's now show that with this, we will find the same result as we did with the approach above. We use $dV = S * dx$ (S is the beam cross-section). Then:

$$\frac{dN}{dt} = \int_0^l n_n \sigma_{ion} v_e n_e(x) S dx = \int_0^l n_n \sigma_{ion} F(x) dx,$$

where $F(x) = v_e n_e(x) S = F_0 \exp\left(-\frac{x}{\lambda_{\text{mfp}}}\right)$ is the flux of beam electrons along x, as calculated above.

$$\Rightarrow \frac{dN}{dt} = n_n \sigma_{ion} F_0 \int_0^l \exp\left(-\frac{x}{\lambda_{\text{mfp}}}\right) dx = n_n \sigma_{ion} \lambda_{\text{mfp}} F_0 \left[1 - \exp\left(-\frac{l}{\lambda_{\text{mfp}}}\right)\right]$$
$$= F_0 \left[1 - \exp\left(-\frac{l}{\lambda_{\text{mfp}}}\right)\right], \quad \text{we used: } n_n \sigma_{ion} \lambda_{\text{mfp}} = 1 \qquad (8)$$

this is consistent with the result found before.

b) In the previous derivation, we assumed that the neutral density is constant. We need to verify this assumption considering the number of neutrals inside the tube[1]: how long could we maintain the discharge for?

The total number of neutrals is:

$$N_n = n_n \cdot \text{volume chambre} = 3.3 \times 10^{19} \cdot 2\pi \approx 2 \times 10^{20}$$

thus we can maintain the discharge for about:

$$\tau_{\text{discharge}} = \frac{N_n}{\frac{dN}{dt}} = \frac{10^{21}}{6 \times 10^{18}} \approx 67 \text{ s} \approx 1 \text{ minute}$$

The assumption is therefore valid for t < 1 minute.

[1]We consider the total volume of the vacuum tube (not only the volume of the beam) because the neutrals are reaching an uniform distribution quickly (accordingly, we can suppose to have the same n_n along x)

Exercise 3

a) The definition of **relative degree of ionization** is

$$\alpha = \frac{n_e}{n_e + n_{Ar}}$$

where $n_{Ar} = N_{Ar}/V$ is the density of neutral Argon atoms and V is the volume of the vacuum chamber.
To evaluate n_{Ar} we can use the ideal gas law:

$$p_{Ar} = n_{Ar} k_B T_{Ar}$$

where $k_B = R/\mathcal{A}$ is the Boltzmann constant and $T_{Ar} = 300$ K (room temperature). This gives

$$n_{Ar} \simeq 3.2 \cdot 10^{18}\ \mathrm{m}^{-3}$$

The degree of ionization with $n_e = 10^{16}\ \mathrm{m}^{-3}$ is:

$$\alpha = \frac{n_e}{\underbrace{n_e + n_{Ar}}_{\approx n_{Ar}}} = \frac{10^{16}}{10^{16} + 3.2 \cdot 10^{18}} \approx 3 \cdot 10^{-3}$$

b) The electron-neutral **collision frequency** is

$$\nu_{en} = n_{Ar} \sigma_{ion} v_{rel}$$

where v_{rel} is the relative velocity between electrons and neutrals and $\sigma_{ion} = 10^3 \pi a_0^2$ is the collision cross-section. Since $m_e \ll m_{Ar}$ and $T_e \gg T_0$ we can assume $v_{rel} \simeq v_e$.

In general, ν_{en} is a function of the electron velocity and, implicitly, $\sigma_{ion} = \sigma_{ion}(v_e)$. In our problem we consider σ_{ion} constant and the mean velocity (see exercise 1)

$$v_e = \langle v \rangle = \sqrt{\frac{8}{\pi}}\, v_{the}$$

where $v_{the} = \sqrt{\frac{T_e}{m_e}}$.

NB: v_e could also be chosen equal to v_{the}.

Numerical result:

$$\nu_{en} = 3.2 \cdot 10^{18}\ \mathrm{m}^{-3}\ 10^3 \pi \underbrace{(5.29 \cdot 10^{-11})^2}_{a_0^2}\ \mathrm{m}^2 \sqrt{\frac{8}{\pi}} \sqrt{\frac{T_e}{m_e}}\ \frac{\mathrm{m}}{\mathrm{s}} \approx 3.3 \cdot 10^7\ \mathrm{s}^{-1}$$

c) **Can we consider this gas a plasma?**

- The Debye length is:

$$\lambda_D = \sqrt{\frac{\varepsilon_0 T_e}{e^2 n_e}} \approx 7430\sqrt{\frac{T_e[\mathrm{eV}]}{n_e\,[\mathrm{m}^{-3}]}} = 7430\sqrt{\frac{3}{10^{16}}} = 0.13 \text{ mm}$$

  Therefore it's necessary to have a plasma size $L_p \gg 0.13$ mm.

- $N_D = \frac{4}{3}\pi\lambda_D^3 n_e \approx 9.2 \cdot 10^4 \gg 1$, so the condition of the plasma parameter $g = N_D^{-1} \ll 1$ is verified.

- To see dynamic collective effects in a plasma (oscillations at the frequency ω_p), we need to have ω_p much bigger than the collision frequency:

$$\omega_p = \sqrt{\frac{e^2 n_e}{m_e \varepsilon_0}} \approx 18\pi\sqrt{n_e\,[\mathrm{m}^{-3}]} = 18\pi\sqrt{10^{16}\ \mathrm{m}^{-3}} = 5.7 \cdot 10^9 \text{ rad/s}$$

  To compare ω_p with ν_{en} we need to convert it in s^{-1}:

$$f_p = \frac{\omega_p}{2\pi} \approx 0.9 \cdot 10^9 \ \mathrm{s}^{-1} > \nu_{en} = 3.3 \cdot 10^7 \ \mathrm{s}^{-1}$$

We can therefore consider this gas as a plasma.

Appendix A : Integration tips

Changing coordinates:

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e^{-(x^2+y^2)} = 2\pi \int_0^{+\infty} r\, dr\, e^{-r^2} =$$

$$= 2\pi \frac{1}{2} \underbrace{\int_0^{+\infty} d\alpha\, e^{-\alpha}}_{1} = \pi$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Derivative:

$$\int_{-\infty}^{+\infty} x^2 e^{-ax^2} dx = \int_{-\infty}^{+\infty} dx \frac{\partial}{\partial(-a)} e^{-ax^2} =$$

$$= \frac{\partial}{\partial(-a)} \int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \frac{\partial}{\partial(-a)} \sqrt{\frac{\pi}{a}} = -\sqrt{\pi} \underbrace{\frac{\partial}{\partial(a)} a^{-1/2}}_{-\frac{1}{2}a^{-3/2}} = \frac{\sqrt{\pi}}{2} a^{-3/2}$$

Using iterative properties:

The Gamma function:

$$\Gamma(x) := \int_0^{+\infty} y^{x-1} e^{-y} dy$$

Properties:

$$\begin{cases} \Gamma(x+1) = x\Gamma(x) \\ \Gamma(1) = 1 \\ \Gamma(1/2) = \sqrt{\pi} \end{cases} \tag{9}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-ax^2} dx = 2\int_0^{+\infty} x^2 e^{-ax^2} dx = \qquad \left| \begin{array}{c} y = ax^2 \\ dy = 2ax\, dx \\ \to dx = \frac{1}{2\sqrt{a}} dy\, y^{-1/2} \end{array} \right|$$

$$\Rightarrow = 2\int_0^{+\infty} \frac{y}{a} e^{-y} \left(dy\, y^{-1/2} \frac{1}{2\sqrt{a}} \right) = a^{-3/2} \int_0^{+\infty} dy\, y^{1/2} e^{-y} =$$

$$= a^{-3/2} \int_0^{+\infty} dy\, y^{3/2-1} e^{-y} = a^{-3/2} \underbrace{\Gamma\left(\frac{3}{2}\right)}_{\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}} = \frac{\sqrt{\pi}}{2} a^{-3/2}$$