Plasma Physics I

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Exercise 1

Consider an unmagnetized plasma described by the Vlasov-Poisson model:

$$\frac{\partial f_{\alpha 1}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\alpha 1}}{\partial \mathbf{x}} + \frac{q_a}{m_a} \mathbf{E}_1 \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\varepsilon_0} \sum_\alpha q_\alpha \int f_{\alpha 1} d^3 v. \tag{2}$$

Under the electrostatic approximation, the electric field is given by $\mathbf{E} = -\nabla \phi$.

Using the Fourier transform applied to the spatial variables and rewriting the derivatives as $\frac{\partial}{\partial \mathbf{x}} = i\mathbf{k}$, we have:

$$\frac{\partial f_{\alpha 1}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} f_{\alpha 1} - i \frac{q_a}{m_a} \phi \, \mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0 \tag{3}$$

$$k^2 \phi = \frac{1}{\varepsilon_0} \sum_\alpha q_\alpha \int f_{\alpha 1} d^3 v. \tag{4}$$

The Laplace transform can be used for the time variable:

$$p \tilde{f}_{\alpha 1} - g_\alpha + i\mathbf{k} \cdot \mathbf{v} \tilde{f}_{\alpha 1} - i \frac{q_\alpha}{m_\alpha} \tilde{\phi} \, \mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0 \tag{5}$$

$$k^2 \tilde{\phi} = \frac{1}{\varepsilon_0} \sum_\alpha q_\alpha \int \tilde{f}_{\alpha 1} d^3 v, \tag{6}$$

where $g_\alpha = \mathcal{F}[f_{\alpha 1}(t=0)] = \mathcal{F}[\delta(\mathbf{x})\delta(\mathbf{v})] = \delta(\mathbf{v})/(2\pi)^3$ for the test particle q_T, otherwise $g_\alpha = 0$. Here $\mathcal{F}[]$ denotes the Fourier transform.

We find $\tilde{f}_{\alpha 1}$ from the eq.(5):

$$\tilde{f}_{\alpha 1} = i \frac{q_a}{m_a} \tilde{\phi} \frac{\mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p + i\mathbf{k} \cdot \mathbf{v}} + \frac{g_\alpha}{p + i\mathbf{k} \cdot \mathbf{v}}. \tag{7}$$

We can now substitute the expression for $\tilde{f}_{\alpha 1}$ on the eq.(6), obtaining:

$$k^2 \tilde{\phi} = \sum_\alpha \frac{q_\alpha}{\varepsilon_0} \int i \frac{q_\alpha}{m_\alpha} \tilde{\phi} \frac{\mathbf{k} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p + i\mathbf{k} \cdot \mathbf{v}} d\mathbf{v} + \sum_\alpha \frac{q_\alpha}{\varepsilon_0} \int d\mathbf{v} \frac{g_\alpha}{p + i\mathbf{k} \cdot \mathbf{v}}$$

and finally:

$$\underbrace{\tilde{\phi}\left(1-\sum_\alpha \frac{iq_\alpha^2}{\varepsilon_0 m_\alpha k^2}\int d\mathbf{v}\frac{\mathbf{k}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p+i\mathbf{k}\cdot\mathbf{v}}\right)}_{D(\mathbf{k},ip)} = \frac{1}{(2\pi)^3}\frac{q_T}{\varepsilon_0 k^2}\underbrace{\int d\mathbf{v}\frac{\delta(\mathbf{v})}{p+i\mathbf{k}\cdot\mathbf{v}}}_{\frac{1}{p}}$$

From the following equation:

$$\tilde{\phi}(\mathbf{k},p) = \frac{1}{(2\pi)^3}\frac{q_T}{\varepsilon_0 k^2}\frac{1}{D(\mathbf{k},ip)\,p}$$

using the inverse Laplace transform we can evaluate the time evolution of ϕ:

$$\phi(\mathbf{k},t) = \frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}\tilde{\phi}(k,p)e^{pt}dp = \sum \mathrm{Res}\Big(\tilde{\phi}(k,p)e^{pt}, p=p_k\Big).$$

We are interested on the pole $p=0$ (which corresponds to a stationary condition, $t\to\infty$), therefore:

$$\phi(\mathbf{k},t) = \frac{1}{(2\pi)^3}\frac{q_T}{\varepsilon_0 k^2\, D(\mathbf{k},p=0)}.$$

To evaluate the term $D(\mathbf{k},p)$ for $p=0$,

$$\begin{aligned}
D(\mathbf{k},p=0) &= 1-\sum_\alpha \frac{iq_\alpha^2}{\varepsilon_0 m_\alpha k^2}\int d\mathbf{v}\frac{\mathbf{k}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}}{p+i\mathbf{k}\cdot\mathbf{v}} \\
&= 1-\sum_\alpha \frac{\omega_{\mathrm{p},\alpha}^2}{n_\alpha k^2}\int_{\mathcal{L}} du\frac{dF_{0,\alpha}}{du}\,\frac{1}{u-\frac{\omega}{k}}\bigg|_{\frac{\omega}{k}=0} \\
&= 1-\sum_\alpha \frac{\omega_{\mathrm{p},\alpha}^2}{nk^2}\left[\mathrm{P.V.}\int \frac{dF_{0\alpha}}{du}\frac{1}{u}du + i\pi\,\frac{dF_{0\alpha}}{du}\bigg|_{u=0}\right] \\
&= 1-\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{nk^2}\mathrm{P.V.}\int du\left(-\frac{u}{v_{\mathrm{th}\alpha}^2}\frac{F_{0\alpha}}{u}\right) \\
&= 1+\frac{1}{(k^2\lambda_{De}^2)}+\frac{1}{(k^2\lambda_{Di}^2)} \\
&= 1+\frac{1}{(k^2\lambda_D^2)}
\end{aligned} \tag{8}$$

where $\frac{1}{\lambda_D^2} = \sum_\alpha \frac{\omega_{p\alpha}^2}{v_{th\alpha}^2}$ and we used $\int_{-\infty}^{\infty} F_{\alpha 0}du = n_\alpha$.

Finally:

$$\phi(\mathbf{k}) = \frac{1}{(2\pi)^3}\frac{q_T}{\varepsilon_0 k^2}\frac{1}{\left(1+\frac{1}{(k\lambda_D)^2}\right)} = \frac{1}{(2\pi)^3}\frac{q_T}{\varepsilon_0}\frac{1}{k^2+k_D^2} \tag{9}$$

where we define $k_D = \lambda_D^{-1}$.

We still need to compute the inverse Fourier transform of $\phi(k)$:

$$\phi(\mathbf{x}) = \int d\mathbf{k}\frac{1}{(2\pi)^3}\frac{q_T}{\varepsilon_0}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2+k_D^2} = \frac{q_T}{\varepsilon_0}\frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2+k_D^2}. \tag{10}$$

The integral can be solved in the spherical coordinate frame (for $\mathbf{k}$):

$$\begin{cases} k_x = k \sin\theta \cos\varphi \\ k_y = k \sin\theta \sin\varphi \\ k_z = k \cos\theta \end{cases}$$

with $d\mathbf{k} = k^2 dk \sin\theta \, d\theta \, d\varphi$.

We choose the direction of $\mathbf{x}$ such that $\mathbf{k} \cdot \mathbf{x} = kr\cos\theta$.

$$\begin{aligned} \phi(\mathbf{x}) &= \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^3} \int_0^\infty dk\, k^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi \frac{e^{ikr\cos\theta}}{k^2 + k_D^2} \\ &= \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^3} \int_0^\infty dk \frac{k^2}{k^2 + k_D^2} \int_0^\pi d\theta \sin\theta e^{ikr\cos\theta} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \end{aligned} \quad (11)$$

but

$$\begin{aligned} \int_0^\pi d\theta \sin\theta e^{\;kr\cos\theta} &= -\int_1^{-1} d(\cos\theta) e^{ikr\cos\theta} = \\ &= \frac{e^{ikr} - e^{-ikr}}{ikr} = \frac{2i\sin(kr)}{ikr} = \frac{2}{kr}\sin(kr) \end{aligned}$$

therefore

$$\phi(\mathbf{x}) = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{2}{r} \int_0^\infty dk \frac{k}{k^2 + k_D^2} \sin(kr). \quad (12)$$

Using the polar notation for the complex quantities, we have $\sin(kr) = \mathrm{Im}(e^{ikr})$:

$$\phi(\mathbf{x}) = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{2}{r} \int_0^\infty dk \frac{\mathrm{Im}\left\{ke^{ikr}\right\}}{k^2 + k_D^2}. \quad (13)$$

Due to the symmetry of the integrated function (even function):

$$\int_0^\infty dk \frac{\mathrm{Im}\left\{ke^{ikr}\right\}}{k^2 + k_D^2} = \frac{1}{2} \int_{-\infty}^\infty dk \frac{\mathrm{Im}\left\{ke^{ikr}\right\}}{k^2 + k_D^2}$$

we can write:

$$\phi(\mathbf{x}) = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{1}{r} \int_{-\infty}^\infty dk \frac{\mathrm{Im}\left\{ke^{ikr}\right\}}{k^2 + k_D^2} = \frac{q_T}{\varepsilon_0} \frac{1}{(2\pi)^2} \frac{1}{r} \mathrm{Im} \left\{ \int_{-\infty}^\infty dk \frac{ke^{ikr}}{k^2 + k_D^2} \right\}. \quad (14)$$

The function $\frac{ke^{ikr}}{k^2 + k_D^2}$ has two poles $k = \pm ik_D$.

To evaluate the integral, we use the residues theorem based on a contour $\mathcal{C}$ that goes along the real axis and then counterclockwise along the semicircle Γ [1](see figure 1).

$$\int_{\mathcal{C}} \frac{ke^{ikr}}{k^2 + k_D^2} = \int_{-\infty}^\infty dk \frac{ke^{ikr}}{k^2 + k_D^2} + \underbrace{\int_\Gamma \frac{ke^{ikr}}{k^2 + k_D^2}}_{=0} \quad (15)$$

[1] if we evaluate the integral clockwise along a semicircle (pole $-ik_D$ bounded by the contour), the integral diverges.

The integral along Γ vanishes because $\mathrm{Im}(k) \to +\infty$. The integral along the contour $\mathcal{C}$ is given by the sum of the residues of the poles bounded by the integration path:

$$\begin{aligned} \int_{\mathcal{C}} \frac{ke^{ikr}}{k^2+k_D^2} &= 2\pi i \mathrm{Res}\left(\frac{ke^{ikr}}{k^2+k_D^2}, k=+ik_D\right) \\ &= \lim_{k\to ik_D} 2\pi i \frac{ke^{ikr}(k-ik_D)}{(k+ik_D)(k-ik_D)} = 2\pi i \frac{ik_D e^{-k_D r}}{2ik_D} = i\pi e^{-r/\lambda_D} \end{aligned} \tag{16}$$

Finally we have:

$$\phi(\mathbf{x}) = \frac{q_T}{4\pi\varepsilon_0}\frac{e^{-r/\lambda_D}}{r} \tag{17}$$

that is the same result we have found at the beginning of the course – see lecture 1, Eq.(1.7).

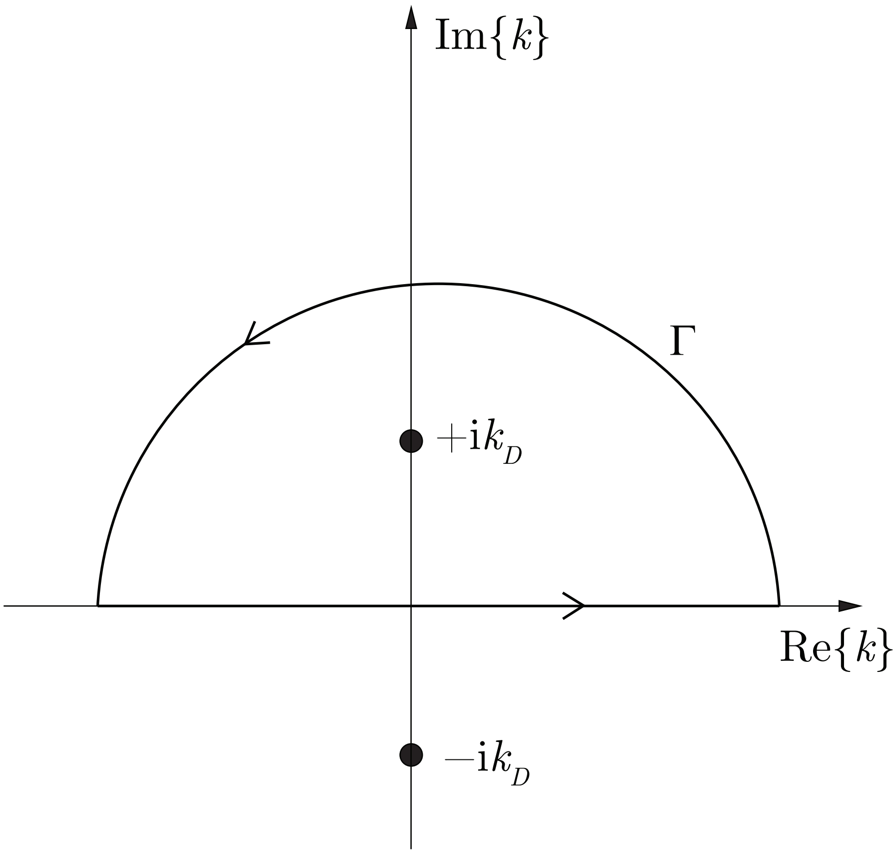

Figure 1: Integration path.