Plasma Physics I

Solution to the Series 11 (November 29, 2025)

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Exercise 1

a) For the electron plasma waves (Langmuir waves), we have $|\omega/k| \gg v_{th,e}$. Therefore we can find the dispersion relation neglecting the contribution from the ion population. To calculate the damping rate of the wave:

$$\gamma = -\frac{\epsilon_i(\omega_r)}{\partial \epsilon_r / \partial \omega_r},$$

we need to evaluate the real part, $\epsilon_r(\omega_r)$, and the imaginary part, $\epsilon_i(\omega_r)$, of $\epsilon(\omega_r, k)$:

$$\begin{cases} \epsilon_r(\omega_r, k) = 1 - \dfrac{\omega_{pe}^2}{n_e k^2} \text{ P.V.} \displaystyle\int du \frac{\frac{dF_{e0}}{du}}{u - \frac{\omega_r}{k}} \\ \epsilon_i(\omega_r, k) = -\pi \dfrac{\omega_{pe}^2}{n_e k^2} \dfrac{dF_{e0}}{du}\Bigg|_{u=\frac{\omega_r}{k}} \end{cases} \tag{1}$$

We have already solved the dispersion relation $\epsilon_r(\omega_r) = 0$ for the Langmuir waves (section 8.2, lecture IX), keeping the first three orders of the expansion in ku/ω:

$$\epsilon_r(\omega_r) \simeq 1 - \frac{\omega_{pe}^2}{\omega_r^2}\left\{1 + 3\,\frac{k^2 v_{th,e}^2}{\omega_r^2}\right\} = 0.$$

For $\omega_r^2 \sim \omega_{pe}^2$, we have found the solution $\omega_r^2 \simeq \omega_{pe}^2 + 3\,k^2 v_{th,e}^2$.

To calculate the derivative $\partial\epsilon_r/\partial\omega_r$, we keep only the dominant term of the expansion (zero-order):

$$\frac{\partial \epsilon_r}{\partial \omega_r} = \frac{\partial}{\omega_r}\left\{1 - \frac{\omega_{pe}^2}{\omega_r^2}\right\} = 2\,\frac{\omega_{pe}^2}{\omega_r^3}. \tag{2}$$

Considering a Maxwellian distribution for the electrons:

$$F_{e0} = \frac{n_e}{\sqrt{2\pi} v_{th,e}} \exp\left\{\left(-\frac{u^2}{2\,v_{th,e}^2}\right)\right\}$$

with $v_{th,e}^2 = T_e/m_e$, we obtain:

$$\epsilon_i(\omega_r) = \sqrt{\frac{\pi}{2}} \frac{\omega_r \omega_{pe}^2}{k^3 v_{th,e}^3} \exp\left\{-\frac{\omega_r^2}{2\,k^2 v_{th,e}^2}\right\}. \tag{3}$$

The damping rate is therefore given by:

$$\gamma = -\sqrt{\frac{\pi}{8}} \frac{\omega_r^4}{k^3 v_{th,e}^3} \exp\left\{-\frac{\omega_r^2}{2\,k^2 v_{th,e}^2}\right\}. \tag{4}$$

We need now to substitute the expression for ω_r. Keeping only the dominant term $\omega \simeq \omega_{pe}$ and considering the correction in the exponential introduced by the third order term $\omega_r^2 \simeq \omega_{pe}^2 + 3k^2 v_{th,e}^2$[1], we find:

$$\gamma = -\sqrt{\frac{\pi}{8}} e^{-3/2} \frac{\omega_{pe}^4}{k^3 v_{th,e}^3} \exp\left\{-\frac{\omega_{pe}^2}{2\,k^2 v_{th,e}^2}\right\}. \tag{5}$$

b) Using the identity $\omega_{pe}/v_{th,e} \equiv \lambda_D^{-1}$, we can write:

$$\gamma = -\sqrt{\frac{\pi}{8}} e^{-3/2} \frac{\omega_{pe}}{(k\lambda_D)^3} \exp\left\{-\frac{1}{2(k\lambda_D)^2}\right\}. \tag{6}$$

The important quantity is therefore the ratio between the Debye length and the wave length. It's easy to verify that $\gamma \to 0$ if $k\lambda_D \to 0$ and that there is a strong damping when $k\lambda_D \sim 1$. Imposing $d\gamma/d(k\lambda_D) = 0$, we can find the condition necessary to have the maximum damping rate $\gamma_{\max}$, that is $k\lambda_D = \sqrt{3}/3$.

c) From part b) we find that:

$$|\frac{\gamma}{\omega_r}| \simeq |\frac{\gamma}{\omega_{pe}}| = |-\sqrt{\frac{\pi}{8}} e^{-3/2} \frac{1}{(k\lambda_D)^3} \exp\left\{-\frac{1}{2(k\lambda_D)^2}\right\}| < |\frac{\gamma_{\max}}{\omega_{pe}}| \simeq 0.16.$$

We thus indeed find that $\gamma/\omega_r \ll 1$

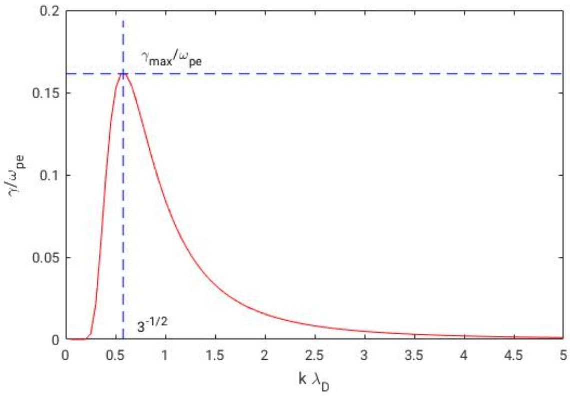

Figure 1: **Damping rate of a Langmuir wave**: plot of the function $f(x) = \frac{\gamma}{\omega_{pe}} = \sqrt{\pi/8} e^{-3/2} \frac{1}{x^3} \exp\{(-\frac{1}{2x^2})\}$.

[1]The term $3\,k^2 v_{th,e}^2$ gives a factor $e^{-3/2} \approx 0.2$.

Exercise 2

Considering the equation:

$$\epsilon(\omega, k) = 1 + \sum_\alpha \frac{e^2}{m_\alpha \epsilon_0 k} \int_{\mathcal{L}} \frac{dF_{0,\alpha}}{du} \frac{1}{\omega - ku} du = 0 \tag{7}$$

for a hydrogen neutral plasma ($n_e = n_i = n_0$, $Z = 1$), with the approximation

$$kv_{th,i} \ll \omega \ll kv_{th,e} \qquad\qquad T_e \gg T_i, \tag{8}$$

it is possible to derive the dispersion relation for the ion-acoustic waves (serie 9, exercise 2).

From the Landau's theory, the damping rate γ is given by:

$$\gamma \equiv -\frac{\epsilon_i(\omega_r, k)}{\partial \epsilon_r(\omega_r, k)/\partial \omega_r}, \tag{9}$$

where we can identify $\epsilon(\omega, k) = \epsilon_r(\omega, k) + i\epsilon_i(\omega, k) \equiv D(\omega, k)$ (electrostatic approximation) and the terms ω_r and γ, that are respectively the real part and the imaginary part of frequency ω:

$$\omega = \omega_r + i\gamma. \tag{10}$$

The eq.(9) is valid only if the absolute value of the damping rate is much smaller then the real part of the frequency ($|\gamma| \ll |\omega_r|$).

Real part of $\epsilon(\omega_r, k)$

The real part of $\epsilon(\omega_r, k)$ is given by:

$$\epsilon_r(\omega_r, k) = 1 + \sum_\alpha \frac{e^2}{m_\alpha \epsilon_0 k} \text{P.V.} \int_{-\infty}^{\infty} \frac{dF_{0,\alpha}}{du} \frac{1}{\omega_r - ku} du \tag{11}$$

Using directly the dispersion relation for the ion-acoustic waves, that we have found for $\omega \simeq \omega_r$ in serie 9 exercise 2, we have:

$$\begin{aligned} \epsilon_r(\omega_r, k) &\simeq 1 - \frac{e^2 n_i}{m_i \varepsilon_0} \frac{1}{\omega_r^2} + \frac{e^2 n_e}{m_e \epsilon_0} \frac{1}{k^2 v_{th,e}^2} \\ &= 1 - \frac{\omega_{pi}^2}{\omega_r^2} + \frac{\omega_{pe}^2}{k^2 v_{th,e}^2}, \end{aligned} \tag{12}$$

that gives, assuming $\lambda \gg \lambda_D$:

$$\frac{\omega_r}{k} \simeq c_s \equiv \sqrt{\frac{T_e}{m_i}}, \tag{13}$$

where c_s is the ion-acoustic speed. The derivative of ϵ_r with respect to ω_r is therefore:

$$\frac{\partial \epsilon_r(\omega_r, k)}{\partial \omega_r} = 2\frac{\omega_{pi}^2}{\omega_r^3} \tag{14}$$

Imaginary part of $\epsilon(\omega_r, k)$

The imaginary part of ϵ is given by:

$$\epsilon_i(\omega_r, k) = -\pi \sum_\alpha \frac{e^2}{m_\alpha \varepsilon_0 k^2} \left. \frac{dF_{0,\alpha}}{du} \right|_{u=\omega_r/k} . \tag{15}$$

For a Maxwellian distribution function, we have:

$$\frac{dF_{0,\alpha}}{du} = -\frac{u}{v_{th,\alpha}^2} F_{0,\alpha}, \tag{16}$$

therefore

$$\epsilon_i(\omega_r, k) = \sqrt{\frac{\pi}{2}} \frac{e^2 n_0 \omega_r}{\varepsilon_0 k^3} \left[\frac{\exp\left(-\frac{\omega_r^2}{2\,k^2 v_{th,e}^2}\right)}{m_e v_{th,e}^3} + \frac{\exp\left(-\frac{\omega_r^2}{2\,k^2 v_{th,i}^2}\right)}{m_i v_{th,i}^3} \right]. \tag{17}$$

Having $\omega_r/k = c_s \ll v_{th,e}$, we find:

$$\exp\left(-\frac{\omega_r^2}{2\,k^2 v_{th,e}^2}\right) \simeq 1. \tag{18}$$

Introducing the definition of electron and ion plasma frequency:

$$\omega_{\mathrm{p},e/i}^2 = \frac{e^2 n_0}{\varepsilon_0 m_{e/i}} \tag{19}$$

and with the substitution $\omega_r/k = c_s$, we can rewrite ϵ_i as:

$$\epsilon_i(\omega_r, k) = \sqrt{\frac{\pi}{2}} \frac{c_s}{k^2} \left[\frac{\omega_{pe}^2}{v_{th,e}^3} + \frac{\omega_{pi}^2}{v_{th,i}^3} \exp\left(-\frac{c_s^2}{2\,v_{th,i}^2}\right) \right]. \tag{20}$$

Damping rate

From the results in the previous sections, eq.(14) and eq.(20), we can evaluate the damping rate using eq.(9):

$$\gamma = -\sqrt{\frac{\pi}{8}} \left[\frac{\omega_{pe}^2 c_s^4}{k^2 v_{th,e}^3} \frac{k^3}{\omega_{pi}^2} + \frac{\omega_{pi}^2 c_s^4}{k^2 v_{th,i}^3} \frac{k^3}{\omega_{pi}^2} \exp\left(-\frac{c_s^2}{2\,v_{th,i}^2}\right) \right]. \tag{21}$$

A possible simplification of the expression for γ is obtained introducing explicitly the thermal velocities and the sound speed. The result is:

$$\gamma = \underbrace{-\sqrt{\frac{\pi}{8}} k c_s \sqrt{\frac{m_e}{m_i}}}_{\gamma_e} \underbrace{- \sqrt{\frac{\pi}{8}} k c_s \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e}{2\,T_i}\right)}_{\gamma_i} \tag{22}$$