Plasma Physics I

Solution to the Series 10 (November 22, 2025)

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Exercise 1

When the electrostatic approximation is valid ($B_1 = 0$), the Faraday equation for the first order terms is:

$$\nabla \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} = 0 \tag{1}$$

and in the Fourier space:

$$\vec{k} \times \vec{E}_1 = 0 \qquad \Rightarrow \qquad \vec{k} \parallel \vec{E}_1 \tag{2}$$

Since we are considering the case with $\theta = 0$, choosing $\vec{B}_0 = B_0 \vec{e}_z$ we have $\vec{k} = (0, 0, k_z)$. The only non-zero component of $\vec{E}_1$ is therefore $E_{1,z}$. We are looking for the dispersion relation from:

$$\begin{pmatrix} -N^2 + \epsilon_1 & -\mathrm{i}\epsilon_2 & 0 \\ \mathrm{i}\epsilon_2 & -N^2 + \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ E_{1,z} \end{pmatrix} = 0. \tag{3}$$

This equation is satisfied for $\epsilon_3 = 0$, that gives:

$$1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} - \frac{\omega_{\mathrm{p}i}^2}{\omega^2} = 0 \Rightarrow \omega^2 = \omega_{\mathrm{pe}}^2 + \omega_{\mathrm{p}i}^2 = \omega_{\mathrm{p}}^2 \tag{4}$$

where $\omega_{\mathrm{p}} \approx \omega_{\mathrm{pe}}$. Therefore, the plasma oscillation is electrostatic.

Further discussion of the electrostatic approximation can be found on p. 77 of the polycopier.

Exercise 2

The *Fokker-Planck* equation includes a collision term that has to be added on the right side of the Vlasov equation. For the uni-dimensional case, that collision term is:

$$\frac{\partial f_t}{\partial t} = \nu \frac{\partial}{\partial w} \left(w f_t + v_{\mathrm{th},f}^2 \frac{\partial f_t}{\partial w} \right) \tag{5}$$

where f_t is the distribution function of the velocity of the *test* particles, w is their velocity and ν is the collision frequency between test and field particles.

Stationary condition means that the left side of the equation above is zero: $\frac{\partial f_t}{\partial t} = 0$. Imposing this condition we obtain:

$$\begin{aligned}
&\frac{\partial}{\partial w}\left[w f_t + v_{\mathrm{th},f}^2 \frac{\partial f_t}{\partial w}\right] = 0 \\
\Rightarrow\quad & w f_t + v_{\mathrm{th},f}^2 \frac{\partial f_t}{\partial w} = \text{const} = A \\
\Rightarrow\quad & \frac{\partial f_t}{\partial w} + \frac{w}{v_{\mathrm{th},f}^2} f_t - \frac{A}{v_{\mathrm{th},f}^2} = 0 \\
\Rightarrow\quad & \frac{\partial f_t}{\partial \xi} + \xi f_t - \frac{A}{v_{\mathrm{th},f}} = 0
\end{aligned} \tag{6}$$

where, to simplify the problem, we changed the variable as follows:

$$\xi = \frac{w}{v_{\mathrm{th},f}} \Rightarrow \frac{\partial}{\partial w} = \frac{1}{v_{\mathrm{th},f}} \frac{\partial}{\partial \xi}. \tag{7}$$

First we show that A has to be $= 0$ in order to obtain physical f_t. To prove this we integrate each term of eq. 6 from $-\infty$ to $+\infty$:

$$\int_{-\infty}^{+\infty} \frac{\partial f_t}{\partial \xi} d\xi = f_t(+\infty) - f_t(-\infty) = 0 \tag{8}$$

$$\int_{-\infty}^{+\infty} \xi f_t d\xi = n_t \cdot \langle \xi \rangle \Rightarrow \text{finite number} \tag{9}$$

where n_t is the density of test particles.

$$\int_{-\infty}^{+\infty} \frac{A}{v_{\mathrm{th},f}} d\xi \to \infty \text{ if } A \neq 0 \tag{10}$$

After showing that $A = 0$, the differential equation for f_t is:

$$\frac{\partial f_t}{\partial \xi} + \xi f_t = 0 \tag{11}$$

this is an integrable equation. It's solution is:

$$f_t = N \cdot \exp\left(-\frac{1}{2}\xi^2\right) = N \cdot \exp\left(-\frac{w^2}{2 v_{\mathrm{th},f}^2}\right) \tag{12}$$

where N is the normalization constant.