C. Theiler, Nov. '25

Motivation for the definition of the pressure tensor in dilute gases and weakly-coupled plasmas

$$\Pi_{ik}^{\alpha} = m_\alpha \int f_\alpha(\vec{x},\vec{v},t)\, v_i v_k \, d^3v$$

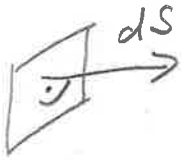
is the <u>momentum flux density tensor</u> for species α. It represents the flux of i-momentum across a surface whose normal vector points along k.

Using $\vec{v} = (\vec{v}-\vec{u}_\alpha+\vec{u}_\alpha)$, with $\vec{u}_\alpha$ the fluid velocity, we can write

$$\Pi_{ik}^{\alpha} = m_\alpha n_\alpha u_{\alpha,i} u_{\alpha,k} + m_\alpha \int f_\alpha(\vec{x},\vec{v},t)\,(v_i-u_{\alpha,i})(v_k-u_{\alpha,k})\, d^3v$$

The first term represents the <u>convection</u> of momentum, the second term is the <u>pressure tensor p_{ik}</u>.

Take a surface element $d\vec{S}$.



Then, $\sum_k \Pi_{ik}^{\alpha} dS_k$ is the i-component of the <u>momentum flux</u> through that element. It can be interpreted as the force exerted on the volume behind the surface element $d\vec{S}$. Examples:

- assume that $d\vec{S}$ is an element of a wall at rest. In that case, we have $\vec{u}_\alpha \cdot d\vec{S} = 0$, and $\sum_k \Pi_{ik}^{\alpha} dS_k = \boxed{\sum_k p_{ik}\, dS_k}$ is the force exerted on $d\vec{S}$. It can have a perpendicular component and a tangential one (friction or viscosity)
- let's assume a surface element $d\vec{S}$ that moves at velocity $\vec{V}$. The <u>momentum flux</u> carried by the particles <u>across that surface</u> is $m_\alpha \int \vec{v}\left[(\vec{v}-\vec{V})\cdot d\vec{S}\right] f_\alpha(\vec{x},\vec{v},t)\, d^3v$

This can be interpreted as a force acting on that element $d\vec{S}$ (or on the volume behind it).

Now, one can show that this integral equals:

$$m_\alpha n_\alpha \vec{u}_\alpha \left[(\vec{u}_\alpha-\vec{V})\cdot d\vec{S}\right] + \underline{\underline{p}}\cdot d\vec{S}$$, with $\underline{\underline{p}}$ the pressure tensor.

So, if $d\vec{S}$ moves with the fluid (assume it's the interface between two fluid elements), then the force on $d\vec{S}$ is just $\boxed{\underline{\underline{p}}\cdot d\vec{S}}$, as it should!