6.7. Propagation perpendicular to $\vec{B}_0$ ($\theta = \pi/2$)

Condition for non-trivial solution becomes

$$\det \begin{pmatrix} -N^2+\varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & -N^2+\varepsilon_3 \end{pmatrix} = 0$$

$$\Rightarrow \quad (-N^2+\varepsilon_3)\cdot\left[\varepsilon_1(-N^2+\varepsilon_1)-\varepsilon_2^2\right] = 0$$

$\rightarrow$ two families of solutions

$$\boxed{\text{Ordinary mode (OM)}}$$

$$(-N^2+\varepsilon_3) = 0$$

In this case, $\vec{E} \parallel \vec{B}_0$ and the dispersion relation reads

$$\frac{k^2c^2}{\omega^2} = 1 - \sum \frac{\omega_{p\alpha}^2}{\omega^2} \approx 1 - \frac{\omega_{pe}^2}{\omega^2}$$

and therefore

$\underline{\omega^2 \simeq \omega_{pe}^2 + k^2c^2}$ $\rightarrow$ Electromagnetic wave in vacuum ($\omega^2 = k^2c^2$) modified by the presence of the plasma ($\omega^2 = k^2c^2 + \omega_{pe}^2$)

ω_{pe} is a <u>cut-off</u> frequency

Extraordinary mode (XM)

$$\left[\varepsilon_1\left(-N^2+\varepsilon_1\right)-\varepsilon_2^2\right]=0$$

In this case, $E_x \neq 0$, $E_y \neq 0$, $E_z = 0$, and the dispersion relation reads

$$N^2 = \frac{\varepsilon_1^2-\varepsilon_2^2}{\varepsilon_1} = \frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1-\varepsilon_2)}{\varepsilon_1} = \frac{\varepsilon_L\,\varepsilon_R}{\varepsilon_1}$$

we have

$$\varepsilon_1 = 1+\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_\alpha^2-\omega^2} = 1+\frac{\omega_{pe}^2}{\Omega_e^2-\omega^2}+\frac{\omega_{pi}^2}{\Omega_i^2-\omega^2}$$

$$= \frac{(\Omega_e^2-\omega^2)(\Omega_i^2-\omega^2)+\omega_{pe}^2(\Omega_i^2-\omega^2)+\omega_{pi}^2(\Omega_e^2-\omega^2)}{(\Omega_e^2-\omega^2)(\Omega_i^2-\omega^2)}$$

$$= \quad \Omega_e \gg \Omega_i;\ \omega_{pe} \gg \omega_{pi}$$

$$\simeq \frac{\omega^4+\omega^2(-\Omega_e^2-\omega_{pe}^2)+\Omega_e^2\Omega_i^2+\omega_{pe}^2\Omega_i^2+\omega_{pi}^2\Omega_e^2}{(\Omega_e^2-\omega^2)(\Omega_i^2-\omega^2)}$$

the numerator has the following roots

$$\omega_{1,2}^2 = \frac{\Omega_e^2+\omega_{pe}^2}{2} \pm \frac{1}{2}\sqrt{\Omega_e^4+2\Omega_e^2\omega_{pe}^2+\omega_{pe}^4-4(\Omega_e^2\Omega_i^2+\omega_{pe}^2\Omega_i^2+\omega_{pi}^2\Omega_e^2)}$$

$\Omega_e \gg \Omega_i$, $\omega_{pe} \gg \omega_{pi}$ $\underline{\underline{\ }}$

$$\simeq \Omega_e^2+\omega_{pe}^2 = \omega_{UH}^2 \quad \text{(upper hybrid resonance)}$$

$$= \frac{(\Omega_e^2+\omega_{pe}^2)}{2}\left[1-\sqrt{1-4\frac{(\Omega_e^2\Omega_i^2+\omega_{pe}^2\Omega_i^2+\omega_{pi}^2\Omega_e^2)}{(\Omega_e^2+\omega_{pe}^2)^2}}\right]$$

use: $1-\sqrt{1-4x} \to 2x$ if $x \to 0$

$$\simeq \frac{(\Omega_e^2\Omega_i^2+\omega_{pe}^2\Omega_i^2+\omega_{pi}^2\Omega_e^2)}{(\Omega_e^2+\omega_{pe}^2)} = \frac{\left(\Omega_i^2\frac{\Omega_e^2}{\omega_{pe}^2}+\Omega_i^2+\omega_{pi}^2\frac{\Omega_e^2}{\omega_{pe}^2}\right)}{1+\Omega_e^2/\omega_{pe}^2}$$

$$= \frac{\Omega_i \Omega_e \left(\overset{-\frac{m_e}{m_i}}{\frac{\Omega_i}{\Omega_e}} \frac{\Omega_e^2}{\omega_{pe}^2} + \cancel{\overset{-\frac{m_e}{m_i}}{\frac{\Omega_i}{\Omega_e}}} + \overset{-1}{\frac{\omega_{pi}^2}{\omega_{pe}^2} \frac{\Omega_e}{\Omega_i}} \right)}{1 + \Omega_e^2/\omega_{pe}^2}$$

$$\simeq \frac{\Omega_i |\Omega_e| \left(1 + \frac{m_e}{m_i} \left(\frac{\Omega_e}{\omega_{pe}} \right)^2 \right)}{1 + \Omega_e^2/\omega_{pe}^2} = \omega_{LH} \quad \text{(lower hybrid resonance)}$$

therefore:

$$\varepsilon_1 = \frac{(\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2)}{(\Omega_e^2 - \omega^2)(\Omega_i^2 - \omega^2)}$$

the dispersion relation for the XM becomes

see expressions for ε_R and ε_L found in Ch. 6.6.

$$N^2 = \frac{\varepsilon_L \varepsilon_R}{\varepsilon_1} = \frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{\cancel{(\Omega_i^2 - \omega^2)}\cancel{(\Omega_e^2 - \omega^2)}} \cdot \overbrace{\frac{\cancel{(\Omega_e^2 - \omega^2)}\cancel{(\Omega_i^2 - \omega^2)}}{(\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2)}}^{= 1/\varepsilon_1}$$

$$\Rightarrow \boxed{N^2 = \frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{(\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2)}}$$

→ the resonances of the XM are: $\omega^2 = \omega_{LH}^2$; $\omega^2 = \omega_{UH}^2$

→ the cutoffs of the XM are: $\omega^2 = \omega_R^2$; $\omega^2 = \omega_L^2$