Fractional quantum Hall effect in a trapping potential:

1) Most of these questions are done in your class.

(a) Anticipating the notations of the next question:

$$H_{\text{Landau}} = \frac{1}{2m}\left(\hat{\Pi}_x^2 + \hat{\Pi}_y^2\right). \tag{1}$$

(b) The computation of the commutator is fairly standard:

$$[\hat{\Pi}_x, \hat{\Pi}_y] = [p_x, \frac{e}{c}A_y] - [p_y, \frac{e}{c}A_x] = -i\frac{e\hbar}{c}(\partial_x A_y - \partial_y A_x) = -i\frac{\hbar eB}{c}. \tag{2}$$

Now, we can define $\hat{P} = \hat{\Pi}_x$ and $\hat{Q} = -\frac{c}{eB}\hat{\Pi}_y$. We then rewrite

$$H_{\text{Landau}} = \frac{1}{2m}\left(\hat{P}^2 + \frac{e^2B^2}{c^2}\hat{Q}^2\right) = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega_c^2\hat{Q}^2, \tag{3}$$

where

$$\omega_c = \frac{eB}{mc}. \tag{4}$$

(c) Working with P and $-Q$ as p_x and x, we recognize the Harmonic oscillator such that $[x, p_x] = i\hbar$. This directly gives us the results.

(d) We compute the two correlators

$$[\hat{X}, \hat{\Pi}_x] = [x, p_x] - \frac{1}{m\omega_c}[\hat{\Pi}_y, \hat{\Pi}_x] = i\hbar - i\hbar\frac{eB}{mc\omega_c} = 0 \tag{5}$$

$$[\hat{X}, \hat{\Pi}_y] = 0 \tag{6}$$

Similarly for $\hat{Y}$. Given the form of the Hamiltonian, we directly have $[\hat{X}, H_{\text{Landau}}] = 0 = [\hat{Y}, H_{\text{Landau}}]$. Just as an example:

$$[\hat{X}, \hat{\Pi}_y^2] = [\hat{X}, \hat{\Pi}_y]\hat{\Pi}_y + \hat{\Pi}_y[\hat{X}, \hat{\Pi}_y] = 0 \tag{7}$$

Finally,

$$[\hat{X}, \hat{Y}] = \frac{1}{m\omega_c}[x, p_x] - \frac{1}{m\omega_c}[p_y, y] - \frac{1}{m^2\omega_c^2}[\hat{\Pi}_x, \hat{\Pi}_y] = \frac{i\hbar}{m\omega_c} \tag{8}$$

(e) Here, we can prove the result in different minimal ways. The simplest is to note that we can formally rewrite the Hamiltonian as

$$H_{\text{Landau}} = \frac{1}{2m}(\hat{\Pi}_x^2 + \hat{\Pi}_y^2) + 0 \times (\hat{X}^2 + \hat{Y}^2). \tag{9}$$

With the previously derived commutation relations, that means we can separately solve the two harmonic oscillators. Let b be the bosonic operator from the second oscillator. As a linear combination of $\hat{X}$ and $\hat{Y}$, it commutes with the Hamiltonian. The general eigenstates are

$$\Phi_{n,m} = \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!m!}}|0\rangle \text{ with energy } \hbar(n+\frac{1}{2})\omega_c. \tag{10}$$

(f) That is where the gauge becomes relevant. Here, two ways offer to us: either we can start from the proposed equations and show that we recover the original Hamiltonian, or write the expression of a as a function of $\hat{\Pi}$. We remember that a must be linear combination of the Π and that

$$\hat{\Pi}_x^2 + \hat{\Pi}_y^2 = 2m\hbar\omega_c(a^\dagger a + \frac{1}{2}) \text{ and } [a, a^\dagger] = 1. \tag{11}$$

Let $a = \alpha_x \hat{\Pi}_x + \alpha_y \hat{\Pi}_y$. That means that

$$|\alpha_x|^2 = |\alpha_y|^2 = \frac{1}{2m\hbar\omega_c} \tag{12}$$

$$\alpha_x^* \alpha_y \hat{\Pi}_x \hat{\Pi}_y + \alpha_y^* \alpha_x \hat{\Pi}_y \hat{\Pi}_x = -\frac{1}{2} \tag{13}$$

$$(\alpha_y \alpha_x^* - \alpha_x \alpha_y^*) i\hbar m\omega_c = 1. \tag{14}$$

The first line is in fact not necessary to solve the problem. The second line imposes $\alpha_x^* \alpha_y$ to be purely imaginary, and the third line then fix the norm. Let $\alpha_x = \frac{i}{\sqrt{2\hbar m\omega_c}}$ and $\alpha_y = -i\alpha_x$. Then

$$a = \frac{1}{\sqrt{2\hbar m\omega_c}}(i\hat{\Pi}_x + \hat{\Pi}_y) = \frac{1}{\sqrt{2\hbar m\omega_c}}(i(-i\hbar\partial_x - \frac{eB}{c}\frac{y}{2}) + (-i\hbar\partial_y + \frac{eB}{c}\frac{x}{2}) \tag{15}$$

$$a = \frac{1}{\sqrt{2\hbar m\omega_c}}(\hbar(\partial_x - i\partial_y) + \frac{m\omega_c}{2}(x+iy)) = \sqrt{2}(\sqrt{\frac{\hbar}{m\omega_c}}\bar{\partial} + \sqrt{\frac{m\omega_c}{\hbar}}\frac{z}{4}). \tag{16}$$

We introduce the elementary length

$$l_B = \sqrt{\frac{\hbar}{m\omega_c}}, \tag{17}$$

which has the correct dimension, and obtain the expression of a we wanted. Note that l_B can be obtained also in the following way. We know that the energy scale is $\hbar\omega_c$, so we can factorize the second half of the Hamiltonian

$$\frac{1}{2}m\omega_c^2 Q^2 = \frac{\hbar\omega_c}{2}\frac{m\omega_c}{\hbar}Q^2. \tag{18}$$

Given that Q is a length, the natural length scale of the problem should be

$$l_B^2 = \frac{\hbar}{m\omega_c}. \tag{19}$$

The Hamiltonian can be rewritten in the suggestive form

$$H_{\text{Landau}} = \hbar\omega_c \left(\frac{l_B^2 \hat{P}^2}{2} + \frac{\hat{Q}^2}{2l_B^2} \right). \tag{20}$$

A similar computation gives the result for b. Note that we have a lot of freedom of choice in the definition of the orbitals. This form is the most convenient on the plane, and for identification with the angular momentum.

(g) We start with $n = m = 0$ The vacuum state verifies

$$a|0\rangle = b|0\rangle = 0, \tag{21}$$

which leads to

$$l_b\overline{\partial}\Phi_{0,0} + \frac{z}{4l_B}\Phi_{0,0} = 0 = l_b\partial\Phi_{0,0} + \frac{z^*}{4l_B}\Phi_{0,0}. \tag{22}$$

The first equation implies

$$\Phi_{0,0}(z, z^*) = e^{-\frac{zz^*}{4l_B^2}} f(z) \tag{23}$$

and the second

$$\Phi_{0,0}(z, z^*) = e^{-\frac{zz^*}{4l_B^2}} g(z^*) \tag{24}$$

Together, we have

$$\Phi_{0,0}(z, z^*) = Ce^{-\frac{zz^*}{4l_B^2}}. \tag{25}$$

The normalization constant is obtained by fixing the norm of $\Phi_{0,0}$ to 1

$$\iint d^2\vec{r}|\Phi_{0,0}|^2 = C^2 \iint d^2\vec{r}e^{-\frac{r^2}{2l_B^2}} = C^2 \times 2\pi l_B^2. \tag{26}$$

To obtain the general form, let s assume that it is correct up for $m \geq 0$.

$$|0, m+1\rangle = \frac{b^\dagger}{\sqrt{m+1}}|0, m\rangle \tag{27}$$

$$\Phi_{0,m+1} = \frac{\sqrt{2}}{\sqrt{m+1}}(-l_B\overline{\partial} + \frac{z}{4l_B})\Phi_{0,m} \tag{28}$$

$$= \frac{\sqrt{2}}{\sqrt{m+1}}(-l_B(-\frac{z}{4l_B}) + \frac{z}{4l_B})\frac{z^m}{\sqrt{2\pi l_B^{2m+2} m! 2^m}} e^{-\frac{zz^*}{4l_B^2}} \tag{29}$$

$$= \frac{z}{\sqrt{2l_B^2(m+1)}}\frac{z^m}{\sqrt{2\pi l_B^{2m+2} m! 2^m}} e^{-\frac{zz^*}{4l_B^2}}. \tag{30}$$

2) We now add the trapping potential $V_t = \frac{1}{2}v_t r^2$.

(a) This computation can be cumbersome if not done appropriately. First we can note that

$$\hat{\Pi}_x^2 = (\hat{\Pi}_x(\alpha) + \frac{\alpha - m\omega_c}{2}y)^2 = \hat{\Pi}_x^2(\alpha) + \frac{(\alpha - m\omega_c)^2}{4}y^2 + (\alpha - m\omega_c)\hat{\Pi}_x y. \tag{31}$$

We used the commutation of $\hat{\Pi}_x$ and y. Then, we can write

$$\hat{Y}(\alpha)^2 = y^2 + \alpha^{-2}\hat{\Pi}_x^2 + 2\alpha^{-1}y\hat{\Pi}_x(\alpha) \Leftrightarrow y\hat{\Pi}_x(\alpha) = \frac{\alpha}{2}(\hat{Y}(\alpha)^2 - y^2 - \alpha^{-2}\hat{\Pi}_x^2). \tag{32}$$

We then obtain:

$$\hat{\Pi}_x^2 = \hat{\Pi}_x^2(\alpha) + \frac{(\alpha - m\omega_c)^2}{4}y^2 + (\alpha - m\omega_c)\frac{\alpha}{2}(\hat{Y}(\alpha)^2 - y^2 - \alpha^{-2}\hat{\Pi}_x^2) \tag{33}$$

$$= (1 - \frac{\alpha - m\omega_c}{2\alpha})\hat{\Pi}_x^2(\alpha) - \frac{(\alpha - m\omega_c)(\alpha + m\omega_c)}{4}y^2 + \frac{\alpha(\alpha - m\omega_c)}{2}\hat{Y}(\alpha)^2 \tag{34}$$

We obtain the same expression interverting x and y and X and Y for $\hat{\Pi}_y$.

(b) We want the term in y^2 obtained in a) to cancel V_t. This gives us:

$$\alpha^2 = 4mv_t + m^2\omega_c^2 \Leftrightarrow \alpha = \sqrt{4mv_t + m^2\omega_c^2}. \tag{35}$$

Fixing α to this value, we obtain

$$\lambda_1 = \lambda_2 = \frac{1}{2m}\frac{\alpha + m\omega_c}{2\alpha} \tag{36}$$

$$\lambda_3 = \lambda_4 = \frac{1}{2m}\frac{\alpha(\alpha - m\omega_c)}{2} \tag{37}$$

(c) We verify that $[\hat{\Pi}_{x/y}, \hat{X}/\hat{Y}] = 0$. We also have:

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -i\hbar\alpha \tag{38}$$

$$[\hat{X}, \hat{Y}] = i\hbar\alpha^{-1} \tag{39}$$

We therefore recognize two independent harmonic oscillators. To get the energy, we just need to remark that

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -i\hbar m\omega_c \tag{40}$$

In the presence of the trapping field, we have:

$$\tilde{m}_c\tilde{\omega}_c = \alpha \text{ and } \frac{1}{2\tilde{m}_c} = \lambda_1 \tag{41}$$

We obtain

$$\tilde{m}_c = m(\frac{\alpha + m\omega_c}{2\alpha})^{-1} = \frac{2m\alpha}{\alpha + m\omega_c} \tag{42}$$

and therefore

$$\tilde{\omega}_c = \frac{\alpha + m\omega_c}{2}. \tag{43}$$

Similarly, for the second oscillator $\tilde{m}_t\tilde{\omega}_t = \alpha^{-1}$

$$\tilde{m}_t = m(\frac{\alpha(\alpha - m\omega_c)}{2})^{-1} = \frac{2m}{\alpha(\alpha - m\omega_c)}. \tag{44}$$

$$\tilde{\omega}_t = \frac{\alpha - m\omega_c}{2}. \tag{45}$$

Note: when the trapping potential disappear, the effective mass $\tilde{m}_t$ diverges, signifying the rigidity of the effective Harmonic oscillator. We also do recover the appropriate values of $\tilde{\omega}_c$ and $\tilde{\omega}_t$ in the limit $\alpha \to m\omega_c$.

(d) By dimensional analysis, we always have

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -i\hbar^2 l_B^{-2} \text{ and } [\hat{X}, \hat{Y}] = il_B'^2. \tag{46}$$

The length scale of both oscillators remains therefore equal here. Note again how dimensional analysis simplify all such computations. We therefore obtain:

$$\tilde{l}_B^{-4} = \frac{\alpha^2}{\hbar^2} = \frac{mv_t}{\hbar^2} + l_B^{-4} \tag{47}$$

We can define the trapping length $l_t^4 = \frac{\hbar^2}{4mv_t}$ to obtain the nice formula:

$$\tilde{l}_B^{-4} = l_t^{-4} + l_B^{-4}. \tag{48}$$

From there, the rest of the computations is the same as before, with

$$\Phi_{0,m} = \frac{z^m}{\sqrt{2\pi\tilde{l}_B^{2m+2}m!2^m}}e^{-\frac{zz^*}{4\tilde{l}_B^2}} \text{ and the energy } \hbar(\frac{\omega_c + \omega_t}{2} + m\omega_t) \tag{49}$$

3) We now consider a system of N electrons in the previous trapping potential.

(a) We want here to be able to say that all electrons prefer to be in the lowest Landau level rather than an orbital with lower angular momentum but in a higher Landau level. This means here that we want

$$\tilde{\omega}_c \gg \tilde{\omega}_t m_{\text{max}}, \tag{50}$$

where max is the largest occupied orbital. Neglecting edge effects, we therefore get

$$\tilde{\omega}_c \gg \tilde{\omega}_t N \nu^{-1}. \tag{51}$$

(b) See course. The Laughlin wave function is

$$\Phi_{1/m} = \prod_{i<j} (z_i - z_j)^m e^{-\sum_j \frac{|z_i|^2}{4l_B^2}}. \tag{52}$$

At filling 1/3, we have $m = 3$. The total angular momentum correspond to the degree of the polynomial $\prod_{i<j}(z_i - z_j)^m$, i.e., $\frac{3N(N-1)}{2}$ (up to a sign, for simplification). Correspondingly, its energy in the trapping potential is

$$E_{1/3} = \hbar \frac{\tilde{\omega}_c}{2} + \hbar \tilde{\omega}_t / \frac{3N(N-1)}{2} + \frac{1}{2}). \tag{53}$$

(c) From the question 2, we know that two particle states with relative momentum 1 have a wavefunction:

$$\Phi(z_1, z_2) = (z_1 - z_2) P(z_1 + z_2) e^{-\sum_j \frac{|z_i|^2}{4l_B^2}}. \tag{54}$$

In a N-particle wavefunction, the part which corresponds to a relative momentum equal to 1 between particle 1 and 2 is

$$(z_1 - z_2) \tilde{P}(z_1 + z_2, z_3, z_4, \dots) e^{-\sum_j \frac{|z_i|^2}{4l_B^2}}. \tag{55}$$

For a wavefunction of the form $P(z_1, z_2, z_3, z_4, \dots) e^{-\sum_j \frac{|z_i|^2}{4l_B^2}}$, we can extract this contribution by noticing

$$\tilde{P}(z_1 + z_2, z_3, z_4, \dots) = \partial_{z_1 - z_2} P(z_1, z_2, z_3, z_4, \dots)_{|z_1 = z_2}, \tag{56}$$

and therefore is non-zero if and only if $P(z_1, z_2, z_3, z_4, \dots)$ vanishes as δz when $z_2 = z_1 + \delta z \to z_1$. By definition, the wavefunction $\Phi_{1/3}$ cancels as δz^3 when $z_1 \to z_2$, and therefore it has no components with a relative momentum equal to 1 when two particles come together. It is therefore a 0 energy state of V_1.

As V_1 is a sum of projectors ($V_1 = \sum_{i<j} V(i,j)$ with $V(i,j)$ a projector, its eigenvalues are always positive. Ψ_3 is therefore a ground state of that Hamiltonian.

(d) What area does the state $\Phi_{1/3}$ approximately occupy? At fixed number of electrons, can we find a more densely packed groundstate of V_1?

(e) In order or $\Phi^P_{1/3}(z_1,\dots)$ to be a fermionic wavefunction, P needs to be a symmetric polynomial. Following the previous question, any such P allows $P(z_1,\dots)\Phi_3$ to be a zero energy groundstate of V_1 as $\Phi^P_{1/3}$ cancels at least as δz^3 when two particles come together.

(f) We trivially obtain

$$E_{1/3,P} = E_{1/3} + \hbar\tilde{\omega}_t d_P \tag{57}$$

(g) We need to evaluate the area covered by a given orbital m in the lowest Landau level. We need to compute $\langle\Phi_{0,m}|r^2|\Phi_{0,m}\rangle$. A straightforward computation leads to:

$$\langle r^2\rangle = \frac{1}{2\pi l_B^{2m+2} m!2^m}\iint r^2 r^{2m} e^{-r^2/2/l_B^2} d^2\vec{r} \tag{58}$$

We can recognize here a computation that we have done before as the integral is exactly the one that we evaluate when normalizing $\Phi_{0,m+1}$, and therefore

$$\langle r^2\rangle = \frac{2\pi l_B^{2m+2+2}(m+1)!2^{m+1}}{2\pi l_B^{2m+2} m!2^m} = 2(m+1)l_B^2. \tag{59}$$

The area covered by the circular orbital is therefore $2\pi(m+1)l_B^2$. For $\Phi_{1/3}$, the largest occupied orbital is $m_{\max} = 3(N-1)$, and therefore the state occupies an area

$$2\pi(3N-2)l_B^2. \tag{60}$$

(h) There is a single symmetric polynomial of degree 1:

$$P_1 = \sum_j z_j. \tag{61}$$

It is the only state with energy $E_{1/3} + \hbar\tilde{\omega}_t$.
For the degree 2, and the energy level $E_{1/3} + 2\hbar\tilde{\omega}_t$, we have two independent choices:

$$P_2^{(1)} = \sum_j z_j^2 \qquad P_2^{(2)} = \sum_{i\neq j} z_i z_j. \tag{62}$$

Let us briefly proof that $P_2^{(1)}$ and $P_2^{(2)}$ are independent. We are looking for $\lambda_1 P_2^{(1)} + \lambda_2 P_2^{(2)} = 0\,\forall\, z_1,\dots$. By deriving twice the expression in the variable z_j, we obtain $2\lambda_1 = 0$. We directly obtain that the only solution is $\lambda_1 = \lambda_2 = 0$ and therefore the two polynomials are linearly indenpendent.

Now we can prove the general result $\Phi^P_{1/3}(z_1,\dots)$ and $\Phi^Q_{1/3}(z_1,\dots)$ are linearly independent iff. P and Q are linearly independent. We propose two simple proofs, which both start in the same way. We are looking for solutions of $\lambda_P\Phi^P_{1/3} + \lambda_Q\Phi^Q_{1/3} = 0\,\forall\, z_1,\dots$. As the gaussian factor is strictly positive, this is equivalent to:

$$\lambda_P P P_{1/3} + \lambda_Q Q P_{1/3} = 0\,\forall\, z_1,\dots \tag{63}$$

Then we can go two ways:

- We restrict ourself to the set D where all z_j are distincts. Then we have that $P_{1/3}$ is always non-zero. We reduce our problem to

$$\lambda_P P + \lambda_Q Q = 0\,\forall\,(z_1,\dots)\in D. \tag{64}$$

D is a dense set in $\mathbb{C}^N$ (and in fact $\mathbb{C}^N - D$ is of measure 0, though it is not relevant here). By continuity of the polynomials, that means that if Eq. (64) is true on D, it is true on $\mathbb{C}^N$, and therefore it amounts to solving

$$\lambda_P P + \lambda_Q Q = 0 \,\forall\, z_1, \dots \tag{65}$$

Hence the equivalence.

- In a much more pedestrian way, we can derive $3(N-1) + \max(d_P, d_Q)$ times the Eq. (63) to show that the polynomial Q and P are either proportional or $\lambda_P = \lambda_Q = 0$ is the only solution.

From this result, we immediately conclude that the second energy level is twice degenerate.

Finally, the third energy level with energy $E_{1/3} + 3\hbar\tilde{\omega}_t$ is three time degenerate with

$$P_3^{(1)} = \sum_j z_j^3,\ P_3^{(2)} = \sum_{i \neq j} z_i z_j^2,\ \text{and}\ P_3^{(3)} = \sum_{i \neq j \neq k} z_i z_j z_k \tag{66}$$

a generating basis.

(i) In a finite system, the admissible momenta are of the form $\frac{2\pi n}{L}$ with n integers. The groundstate of the toy model is therefore the vacuum state 0. The first excited level is not degenerate: $b^\dagger_{n=1}|0\rangle$ with energy $\frac{2\pi}{L}$. The second excited level is twice degenerate: $\frac{(b^\dagger_{n=1})^2}{\sqrt{2}}|0\rangle$ and $b^\dagger_{n=2}|0\rangle$ with energy $\frac{4\pi}{L}$. The third excited level is three-times degenerate: $\frac{(b^\dagger_{n=1})^3}{\sqrt{6}}|0\rangle$, $b^\dagger_{n=1}b^\dagger_{n=2}|0\rangle$ and $b^\dagger_{n=3}|0\rangle$ with energy $\frac{6\pi}{L}$. In fact, the low energy spectrum of this toy model is identical to the one we previously computed. This is not a random happenstance: the edge mode of the Laughlin state is exactly a chiral bosonic edge mode. The excitations of we studied in V_1 indeed correspond to edge excitations.