1 Problem I

1.1 Preliminary question

$$
\begin{aligned}
[S^z, S^\pm] &= [S^z, S^x] \pm i[S^z, S^y] = iS^y \pm i(-iS^x) = \pm S^\pm && (1)\\
[S^+, S^-] &= [S^x, -iS^y] + [iS^y, S^x] = 2S^z && (2)
\end{aligned}
$$

For a spin 1/2, $\vec{S}^2 = 3/4$.

1.2 Bosonic approach

1) Assuming bosonic commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$, it is straightforward that spin operators acting on different sites commute. We define the operator $n_i = a_i^\dagger a_i$, which obeys $[n_i, a_i] = -a_i$ and $[n_i, a_i^\dagger] = a_i^\dagger$.

$$
\begin{aligned}
[S_i^z, S_i^+] &= S_i^z S_i^+ - \sqrt{1-n_i}\,a_i(1/2 - n_i) && (3)\\
&= S_i^z S_i^+ - \sqrt{1-n_i}\Big(\underbrace{[a_i, 1/2 - n_i]}_{-a_i} + (1/2 - n_i)a_i\Big) && (4)\\
&= S_i^z S_i^+ + \sqrt{1-n_i}\,a_i - S_i^z S_i^+ && (5)\\
&= S_i^+ && (6)
\end{aligned}
$$

$$
\begin{aligned}
[S_i^z, S_i^-] &= (1/2 - n_i)a_i^\dagger\sqrt{1-n_i} - S_i^- S_i^z && (7)\\
&= \Big(\underbrace{[1/2 - n_i, a_i^\dagger]}_{-a_i^\dagger} + a_i^\dagger(1/2 - n_i)\Big)\sqrt{1-n_i} - S_i^- S_i^z && (8)\\
&= -S_i^- + a_i^\dagger\sqrt{1-n_i}(1/2 - n_i) - S_i^- S_i^z && (9)\\
&= -S_i^- && (10)
\end{aligned}
$$

$$
\begin{aligned}
[S_i^+, S_i^-] &= \sqrt{1-n_i}\,a_i a_i^\dagger\sqrt{1-n_i} - a_i^\dagger(1-n_i)a_i && (11)\\
&= \sqrt{1-n_i}\Big(\underbrace{[a_i, a_i^\dagger] + a_i^\dagger a_i}_{1+n_i}\Big)\sqrt{1-n_i} - a_i^\dagger\Big(\underbrace{[1-n_i, a_i]}_{a_i} + a_i(1-n_i)\Big) && (12)\\
&= (1+n_i)(1-n_i) - n_i - n_i(1-n_i) && (13)\\
&= 2S_i^z && (14)
\end{aligned}
$$

Finally we check the norm $\vec{S}_i^2$:

$$
\begin{aligned}
S_i^z S_i^z &= (1/2 - n_i)^2 = 1/4 - n_i + n_i^2 && (15)\\
S_i^+ S_i^- &= (1+n_i)(1-n_i) = 1 - n_i^2 && (16)\\
S_i^- S_i^+ &= n_i + n_i(1-n_i) = 2n_i - n_i^2 && (17)\\
& && (18)
\end{aligned}
$$

and we conclude $\vec{S}_i \cdot \vec{S}_i = S_i^z S_i^z + \frac{1}{2}(S_i^+ S_i^- + S_i^- S_i^+) = 3/4$.

2) In an antiferromagnet, spins point in a different direction on the two sublattices, therefore on sublattice B one must consider the operators:

$$\begin{cases} S_j^z = -1/2 + a_j^\dagger a_j \\ S_j^+ = a_j^\dagger \sqrt{1-n_j} \\ S_j^- = \sqrt{1-n_j} a_j \end{cases} \tag{19}$$

3)

$$\begin{aligned} \mathcal{H} = J \sum_{\langle i,j \rangle} &(1/2 - a_i^\dagger a_i)(-1/2 + a_j^\dagger a_j) \\ &+ 1/2\Big(\sqrt{1-n_i} a_i \sqrt{1-n_j} a_j + a_i^\dagger \sqrt{1-n_i} a_j^\dagger \sqrt{1-n_j}\Big) \end{aligned} \tag{20}$$

Keeping only terms with at most two operators, one gets:

$$\mathcal{H} \approx J/2 \sum_{\langle i,j \rangle} a_j^\dagger a_j + a_i^\dagger a_i - 1/2 + a_i a_j + a_i^\dagger a_j^\dagger \tag{21}$$

4) Defining $a_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_i e^{-i\mathbf{k}\cdot\mathbf{r}_i} a_i$ and $a_{\mathbf{k}}^\dagger = \frac{1}{\sqrt{N}} \sum_i e^{\mathbf{k}\cdot\mathbf{r}_i} a_i^\dagger$, we have

$$\sum_i a_i^\dagger a_i = \frac{1}{N} \sum_i \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} e^{i\mathbf{k}_1\cdot\mathbf{r}_i} a_{\mathbf{k}_1}^\dagger e^{-i\mathbf{k}_2\cdot\mathbf{r}_i} a_{\mathbf{k}_2} \tag{22}$$

$$= \frac{1}{N} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2} \underbrace{\sum_i e^{i(\mathbf{k}_1 - \mathbf{k}_2)}}_{N\delta_{\mathbf{k}_1,\mathbf{k}_2}} \tag{23}$$

$$= \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \tag{24}$$

Considering only nearest neighbors, we sum over $\vec{\tau} = a\hat{x}, a\hat{y}$

$$\sum_{\langle i,j \rangle} a_i a_j = \frac{1}{N} \sum_i \sum_{\vec{\tau}} \sum_{\vec{k}_1} \sum_{\vec{k}_2} e^{i\vec{k}_1\cdot\vec{r}_i} a_{\vec{k}_1} e^{i\vec{k}_2\cdot(\vec{r}_i + \vec{\tau})} a_{\vec{k}_2} \tag{25}$$

$$= \frac{1}{N} \sum_{\vec{k}_2} \sum_{\vec{\tau}} e^{i\vec{k}_2\cdot\vec{\tau}} \sum_{\vec{k}_1} a_{\vec{k}_1} a_{\vec{k}_2} \underbrace{\sum_i e^{i(\vec{k}_1 + \vec{k}_2)\cdot\vec{r}_i}}_{N\delta_{\vec{k}_1,-\vec{k}_2}} \tag{26}$$

$$= \sum_{\vec{k}} \sum_{\vec{\tau}} e^{i\vec{k}\cdot\vec{\tau}} a_{\vec{k}} a_{-\vec{k}} \tag{27}$$

$$= \sum_{\vec{k}} (\cos k_x + \cos k_y) a_{\vec{k}} a_{-\vec{k}} \tag{28}$$

and similarly $\sum_{\langle i,j \rangle} a_i^\dagger a_j^\dagger = \sum_{\vec{k}} (\cos k_x + \cos k_y) a_{\vec{k}}^\dagger a_{-\vec{k}}^\dagger$
We end up with:

$$\mathcal{H}_2 = -NJ/2 + J \sum_{\vec{k}} 2a_{\vec{k}}^\dagger a_{\vec{k}} + (\cos k_x + \cos k_y)\Big(a_{\vec{k}}^\dagger a_{-\vec{k}}^\dagger + a_{\vec{k}} a_{-\vec{k}}\Big) \tag{29}$$

5) We define the bosonic operators $\alpha_{\vec{k}}$ and $\alpha_{\vec{k}}^\dagger$ as

$$\alpha_{\vec{k}} = u_{\vec{k}} a_{\vec{k}} + v_{\vec{k}} a_{-\vec{k}}^\dagger \tag{30}$$

$$\alpha_{\vec{k}}^\dagger = u_{\vec{k}} a_{\vec{k}}^\dagger + v_{\vec{k}} a_{-\vec{k}} \tag{31}$$

$$\tag{32}$$

with real coefficients $u_{\vec{k}}, v_{\vec{k}}$ fulfilling $u_{\vec{k}}^2 - v_{\vec{k}}^2 = 1$. We want to diagonalize the quadratic Hamiltonian (**??**) with these operators, i.e.

$$\mathcal{H}_2 = E_0 + \sum_{\vec{k}} \omega_{\vec{k}} \alpha^\dagger_{\vec{k}} \alpha_{\vec{k}} \tag{33}$$

We look at the commutators:

$$\begin{aligned}
[\mathcal{H}_2, \alpha_{\vec{k}}] &= -\omega_{\vec{k}} \alpha_{\vec{k}} \qquad (34)\\
&= u_{\vec{k}} [\mathcal{H}_2, a_{\vec{k}}] + v_{\vec{k}} [\mathcal{H}_2, -a^\dagger_{-\vec{k}}] \qquad (35)
\end{aligned}$$

$$\begin{aligned}
[\mathcal{H}_2, a_{\vec{k}}] &= 2J[a^\dagger_{\vec{k}} a_{\vec{k}}, a_{\vec{k}}] + J(\cos k_x + \cos k_y)\Big([a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}}, a_{\vec{k}}] + \underbrace{[a_{\vec{k}} a_{-\vec{k}}, a_{\vec{k}}]}_{0}\Big) \qquad (36)\\
&= -2Ja_{\vec{k}} + J(\cos k_x + \cos k_y)(a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} a_{\vec{k}} - \underbrace{a_{\vec{k}} a^\dagger_{\vec{k}}}_{1+a^\dagger_{\vec{k}} a_{\vec{k}}} a^\dagger_{-\vec{k}}) \qquad (37)\\
&= 2Ja_{\vec{k}} - J(\cos k_x + \cos k_y) a^\dagger_{-\vec{k}} \qquad (38)
\end{aligned}$$

$$\begin{aligned}
[\mathcal{H}_2, a^\dagger_{-\vec{k}}] &= 2J[a^\dagger_{-\vec{k}} a_{-\vec{k}}, a^\dagger_{-\vec{k}}] + J(\cos k_x + \cos k_y)\Big(\underbrace{[a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}}, a^\dagger_{-\vec{k}}]}_{0} + [a_{\vec{k}} a_{-\vec{k}}, a^\dagger_{-\vec{k}}]\Big) \qquad (39)\\
&= 2Ja^\dagger_{-\vec{k}} + J(\cos k_x + \cos k_y)(a_{\vec{k}} \underbrace{a_{-\vec{k}} a^\dagger_{-\vec{k}}}_{1+a^\dagger_{-\vec{k}} a_{-\vec{k}}} - a^\dagger_{-\vec{k}} a_{\vec{k}} a_{-\vec{k}}) \qquad (40)\\
&= 2Ja^\dagger_{-\vec{k}} + J(\cos k_x + \cos k_y) a_{\vec{k}} \qquad (41)
\end{aligned}$$

We can rewrite (**??**):

$$\begin{aligned}
[\mathcal{H}_2, \alpha_{\vec{k}}] &= -u_{\vec{k}}(2Ja_{\vec{k}} - J(\cos k_x + \cos k_y) a^\dagger_{-\vec{k}}) + v_{\vec{k}}(2Ja^\dagger_{-\vec{k}} + J(\cos k_x + \cos k_y) a_{\vec{k}}) \qquad (42)\\
&= J(-2u_{\vec{k}} + (\cos k_x + \cos k_y)) a_{\vec{k}} + J(2v_{\vec{k}} + (\cos k_x + \cos k_y)) a^\dagger_{-\vec{k}} \qquad (43)
\end{aligned}$$

We arrive to the eigenvalue problem:

$$\begin{pmatrix} -2J & -J(\cos k_x + \cos k_y) \\ J(\cos k_x + \cos k_y) & 2J \end{pmatrix} \begin{pmatrix} u_{\vec{k}} \\ v_{\vec{k}} \end{pmatrix} = -\omega_{\vec{k}} \begin{pmatrix} u_{\vec{k}} \\ v_{\vec{k}} \end{pmatrix} \tag{44}$$

which admits a solution if the determinant vanishes:

$$(-\omega_{\vec{k}} + 2J)(-\omega_{\vec{k}} - 2J) + J^2(\cos k_x + \cos k_y)^2 = 0 \tag{45}$$

and we conclude $\omega_{\vec{k}} = J\sqrt{4 - (\cos k_x + \cos k_y)^2}$. For small impulsion, we obtain a linear dispersion.

1.3 Fermionic approach

1) We define $n_{i\sigma} = c^\dagger_{i\sigma}c_{i\sigma}$.

$$\begin{aligned}
[S_i^z, S_i^+] &= \frac{1}{2}(c^\dagger_{i\uparrow}c_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow})c^\dagger_{i\uparrow}c_{i\downarrow} - c^\dagger_{i\uparrow}c_{i\downarrow}\frac{1}{2}(c^\dagger_{i\uparrow}c_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow}) && (46)\\
&= \frac{1}{2}c^\dagger_{i\uparrow}c_{i\uparrow}c^\dagger_{i\uparrow}c_{i\downarrow} + \frac{1}{2}c^\dagger_{i\uparrow}c_{i\downarrow}c^\dagger_{i\downarrow}c_{i\downarrow} && (47)\\
&= \frac{1}{2}c^\dagger_{i\uparrow}(\{c^\dagger_{i\uparrow}, c_{i\uparrow}\} - c^\dagger_{i\uparrow}c_{i\uparrow})c_{i\downarrow} + \frac{1}{2}c^\dagger_{i\uparrow}(\{c^\dagger_{i\downarrow}, c^\dagger_{i\downarrow}\} - c^\dagger_{i\downarrow}c_{i\downarrow})c_{i\downarrow} && (48)\\
&= \frac{1}{2}c^\dagger_{i\uparrow}c_{i\downarrow} + \frac{1}{2}c^\dagger_{i\uparrow}c_{i\downarrow} = S_i^+ && (49)
\end{aligned}$$

Similarly by swapping ↑ and ↓ we get $[S_i^z, S_i^-] = -S_i^-$.

$$\begin{aligned}
[S_i^+, S_i^-] &= c^\dagger_{i\uparrow}\underbrace{c_{i\downarrow}c^\dagger_{i\downarrow}}_{1-n_{i\downarrow}}c_{i\uparrow} - c^\dagger_{i\downarrow}\underbrace{c_{i\uparrow}c^\dagger_{i\uparrow}}_{1-n_{i\uparrow}c_{i\uparrow}}c_{i\downarrow} && (50)\\
&= c^\dagger_{i\uparrow}c_{i\uparrow} - n_{i\downarrow}n_{i\uparrow} + n_{i\downarrow}n_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow} = 2S_i^z && (51)
\end{aligned}$$

Finally we check the norm $\vec{S}_i^2$:

$$S_i^zS_i^z = \frac{1}{4}(n_{i\uparrow} - n_{i\downarrow})^2 = \frac{1}{4}(n_{i\uparrow}^2 + n_{i\downarrow}^2 - 2n_{i\uparrow}n_{i\downarrow}) \tag{52}$$

We note that $n_{i\sigma}$ is a projector that obeys $n_{i\sigma}^2 = n_{i\sigma}$, and therefore $n_{i\uparrow}n_{i\downarrow} = n_{i\uparrow}(1 - n_{i\uparrow}) = n_{i\uparrow} - n_{i\uparrow} = 0$, therefore $S_i^zS_i^z = 1/4$. Using this identity, we also see that $S_i^+S_i^- = n_{i\uparrow}$ and $S_i^-S_i^+ = n_{i\downarrow}$ so that $S_i^+S_i^- + S_i^-S_i^+ = 1$ and we conclude $\vec{S}_i \cdot \vec{S}_i = S_i^zS_i^z + \frac{1}{2}(S_i^+S_i^- + S_i^-S_i^+) = 3/4$.

2)a.

$$\begin{aligned}
\chi^\dagger_{ij}\chi_{ij} &= (c^\dagger_{j\uparrow}c_{i\uparrow} + c^\dagger_{j\downarrow}c_{i\downarrow})(c^\dagger_{i\uparrow}c_{j\uparrow} + c^\dagger_{i\downarrow}c_{j\downarrow}) && (53)\\
&= c^\dagger_{j\uparrow}\underbrace{c_{i\uparrow}c^\dagger_{i\uparrow}}_{1-n_{i\uparrow}}c_{j\uparrow} + c^\dagger_{j\uparrow}\underbrace{c_{i\uparrow}c^\dagger_{i\downarrow}}_{-S_i^-}c_{j\downarrow} + c^\dagger_{j\downarrow}\underbrace{c_{i\downarrow}c^\dagger_{i\uparrow}}_{-S_i^+}c_{j\uparrow} + c^\dagger_{j\downarrow}\underbrace{c_{i\downarrow}c^\dagger_{i\downarrow}}_{1-n_{i\downarrow}}c_{j\downarrow} && (54)\\
&= n_{j\uparrow} - n_{j\uparrow}n_{i\uparrow} - S_j^+S_i^- - S_j^-S_i^+ + n_{j\downarrow} - n_{i\downarrow}n_{j\downarrow} && (55)
\end{aligned}$$

Using $n_{j\uparrow} + n_{j\downarrow} = 1$ and

$$\begin{aligned}
S_i^jS_j^z &= \frac{1}{4}(n_{i\uparrow}n_{j\uparrow} - \underbrace{n_{i\uparrow}}_{1-n_{i\downarrow}}n_{j\downarrow} - \underbrace{n_{i\downarrow}}_{1-n_{i\uparrow}}n_{j\uparrow} + n_{i\downarrow}n_{j\downarrow}) && (56)\\
&= \frac{1}{4}(2n_{i\uparrow}n_{j\uparrow} + 2n_{i\downarrow}n_{j\downarrow} - 1) && (57)\\
&\Rightarrow -2S_i^jS_j^z + 1/2 = -n_{i\uparrow}n_{j\uparrow} - n_{i\downarrow}n_{j\downarrow} + 1 && (58)
\end{aligned}$$

we conclude

$$\chi^\dagger_{ij}\chi_{ij} = \frac{1}{2} - 2S_i^jS_j^z - S_j^+S_i^- - S_j^-S_i^+ = \frac{1}{2} - 2\vec{S}_i \cdot \vec{S}_j \tag{59}$$

2)b.

$$\begin{aligned}
\mathcal{H} &= J\sum_{\langle i,j\rangle}\vec{S}_i \cdot \vec{S}_j = J\sum_{\langle i,j\rangle} -\frac{1}{2}\chi^\dagger_{ij}\chi_{ij} + \frac{1}{4}\\
&= \frac{NJ}{2} - \frac{J}{2}\sum_{\langle i,j\rangle}\chi^\dagger_{ij}\chi_{ij} && (60)\\
& && (61)
\end{aligned}$$

3) In the mean-field approximation

$$\mathcal{H}_{MF} = \frac{NJ}{2} - \frac{J}{2} \sum_{\langle i,j\rangle} \left\langle \chi_{ij}^{\dagger} \right\rangle \chi_{ij} + \chi_{ij}^{\dagger} \left\langle \chi_{ij} \right\rangle - \left\langle \chi_{ij}^{\dagger} \right\rangle \left\langle \chi_{ij} \right\rangle \tag{62}$$

4)a. We have 4 inequivalent bonds, the Hamiltonian is defined on a 1 × 2 plaquette, meaning if we shift the system by 2 sites we are back to the initial configuration, while shifting by 1 site exchanges $\chi_1 \leftrightarrow \chi_3$ and $\chi_2 \leftrightarrow \chi_4$.

4)b. Considering the 4 neighbors of a site A at position $\vec{r} - \frac{a}{2}\hat{x}$, we get $\vec{s} \in \{\vec{r},\ \vec{r} - \vec{R}_1,\ \vec{r} + \vec{R}_2,\ \vec{r} - \vec{R}_1 + \vec{R}_2\}$

4)c. We will sum over all sites A, we need to look whether $a^{\dagger}_{\vec{r}\sigma}$ or $b^{\dagger}_{\vec{r}\sigma}$ appears in the definition of χ_i (depends whether $\vec{r}_i$ is the 1st or the 2nd index).

$$\begin{aligned}
\text{right}/\vec{s} = \vec{r} &: \chi_1^{*} \tag{63}\\
\text{down}/\vec{s} = \vec{r} - \vec{R}_1 &: \chi_2 \tag{64}\\
\text{left}/\vec{s} = \vec{r} - \vec{R}_1 + \vec{R}_2 &: \chi_3^{*} \tag{65}\\
\text{up}/\vec{s} = \vec{r} + \vec{R}_2 &: \chi_4 \tag{66}\\
& \tag{67}
\end{aligned}$$

4)d. Dropping the constants, we get:

$$\begin{aligned}
\mathcal{H}_{MF} = -\frac{J}{2} \sum_{\vec{r},\sigma} & \chi_1^{*} a^{\dagger}_{\vec{r}\sigma} b_{\vec{r}\sigma} + \chi_1 b^{\dagger}_{\vec{r}\sigma} a_{\vec{r}\sigma} \\
& + \chi_2 a^{\dagger}_{\vec{r}\sigma} b_{\vec{r}-\vec{R}_1\sigma} + \chi_2^{*} b^{\dagger}_{\vec{r}-\vec{R}_1\sigma} a_{\vec{r}\sigma} \\
& + \chi_3^{*} a^{\dagger}_{\vec{r}\sigma} b_{\vec{r}-\vec{R}_1+\vec{R}_2\sigma} + \chi_3 b^{\dagger}_{\vec{r}-\vec{R}_1+\vec{R}_2\sigma} a_{\vec{r}\sigma} \\
& + \chi_4 a^{\dagger}_{\vec{r}\sigma} b_{\vec{r}+\vec{R}_2\sigma} + \chi_4^{*} b^{\dagger}_{\vec{r}+\vec{R}_2\sigma} a_{\vec{r}\sigma}
\end{aligned} \tag{68}$$

4)e.

$$\begin{aligned}
\mathcal{H}_{MF} = -\frac{J}{2N} \sum_{\vec{r},\sigma} \sum_{\vec{q}_1} \sum_{\vec{q}_2} & e^{i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} \chi_1^{*} a^{\dagger}_{\vec{q}_1\sigma} b_{\vec{q}_2\sigma} + e^{-i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} \chi_1 b^{\dagger}_{\vec{q}_2\sigma} a_{\vec{q}_1\sigma} \\
& + e^{i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} e^{-i\vec{q}_2\cdot\vec{R}_1} \chi_2 a^{\dagger}_{\vec{q}_1\sigma} b_{\vec{q}_2\sigma} + e^{-i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} e^{i\vec{q}_2\cdot\vec{R}_1} \chi_2^{*} b^{\dagger}_{\vec{q}_2\sigma} a_{\vec{q}_1\sigma} \\
& + e^{i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} e^{i\vec{q}_2\cdot(-\vec{R}_1+\vec{R}_2)} \chi_3^{*} a^{\dagger}_{\vec{q}_1\sigma} b_{\vec{q}_2\sigma} + e^{-i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} e^{-i\vec{q}_2\cdot(-\vec{R}_1+\vec{R}_2)} \chi_3 b^{\dagger}_{\vec{q}_2\sigma} a_{\vec{q}_1\sigma} \\
& + e^{i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} e^{i\vec{q}_2\cdot\vec{R}_2} \chi_4 a^{\dagger}_{\vec{q}_1\sigma} b_{\vec{q}_2\sigma} + e^{-i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} e^{-i\vec{q}_2\cdot\vec{R}_2} \chi_4^{*} b^{\dagger}_{\vec{q}_2\sigma} a_{\vec{q}_1\sigma}
\end{aligned} \tag{69}$$

We apply $\sum_{\vec{r}} e^{i(\vec{q}_2-\vec{q}_1)\cdot\vec{r}} = N\delta_{\vec{q}_1,\vec{q}_2}$:

$$\begin{aligned}
\mathcal{H}_{MF} = -\frac{J}{2} \sum_{\sigma} \sum_{\vec{q}} & \chi_1^{*} a^{\dagger}_{\vec{q}\sigma} b_{\vec{q}\sigma} + \chi_1 b^{\dagger}_{\vec{q}\sigma} a_{\vec{q}\sigma} \\
& + e^{-i\vec{q}\cdot\vec{R}_1} \chi_2 a^{\dagger}_{\vec{q}\sigma} b_{\vec{q}\sigma} + e^{i\vec{q}\cdot\vec{R}_1} \chi_2^{*} b^{\dagger}_{\vec{q}\sigma} a_{\vec{q}\sigma} \\
& + e^{i\vec{q}\cdot(-\vec{R}_1+\vec{R}_2)} \chi_3^{*} a^{\dagger}_{\vec{q}\sigma} b_{\vec{q}\sigma} + e^{-i\vec{q}\cdot(-\vec{R}_1+\vec{R}_2)} \chi_3 b^{\dagger}_{\vec{q}\sigma} a_{\vec{q}\sigma} \\
& + e^{i\vec{q}\cdot\vec{R}_2} \chi_4 a^{\dagger}_{\vec{q}\sigma} b_{\vec{q}\sigma} + e^{-i\vec{q}\cdot\vec{R}_2} \chi_4^{*} b^{\dagger}_{\vec{q}\sigma} a_{\vec{q}\sigma}
\end{aligned} \tag{70}$$

Finally, we obtain

$$\mathcal{H}_{MF} = \sum_{\vec{q},\sigma} F_{\vec{q}} a^{\dagger}_{\vec{q}\sigma} b_{\vec{q}\sigma} + F^{*}_{\vec{q}} b^{\dagger}_{\vec{q}\sigma} a_{\vec{q}\sigma} \tag{71}$$

with

$$F_{\vec{q}} = -\frac{J}{2}\left(\chi_1^{*} + e^{-i\vec{q}\cdot\vec{R}_1}\chi_2 + e^{i\vec{q}\cdot(-\vec{R}_1+\vec{R}_2)}\chi_3^{*} + e^{i\vec{q}\cdot\vec{R}_2}\chi_4\right) \tag{72}$$

4)f. Consider the occupation basis $\{\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle\}$:

$$\begin{align} H\left|00\right\rangle &= 0 \tag{73}\\ H\left|01\right\rangle &= -F\left|10\right\rangle \tag{74}\\ H\left|10\right\rangle &= -F^{*}\left|01\right\rangle \tag{75}\\ H\left|11\right\rangle &= 0 \tag{76}\end{align}$$

Note that the sign is a convention and can be absorbed in the phase θ. We have 2 zero eigenvalues and a matrix $m = \begin{pmatrix} 0 & -F \\ -F^{*} & 0 \end{pmatrix}$. Its eigenvalues λ_1 and λ_2 are real and obey

$$\begin{align} \operatorname{Tr} m &= \lambda_1 + \lambda_2 = 0 \tag{77}\\ \det m &= \lambda_1\lambda_2 = -|F|^2 \tag{78}\end{align}$$

therefore $\lambda_{1/2} = \pm|F|$. $x_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}$ and $x_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\theta} \end{pmatrix}$ are associated eigenvectors.

4)g. The spectrum is made of $\pm|F_{\vec{q}}|$, which give 2 different branches. This is a consequence of the unit cell containing 2 sites.

$$\epsilon_{\vec{q}} = \pm\frac{J}{2}\left|\chi_1^{*} + e^{-i\vec{q}\cdot\vec{R}_1}\chi_2 + e^{i\vec{q}\cdot(-\vec{R}_1+\vec{R}_2)}\chi_3^{*} + e^{i\vec{q}\cdot\vec{R}_2}\chi_4\right| \tag{79}$$

4)h. In this case,

$$\begin{align} F_{\vec{q}} &= -\frac{J\chi}{2}\left(e^{-i\pi/4} + e^{-iq_1}e^{i\pi/4} + e^{-iq_1}e^{iq_2}e^{-i\pi/4} + e^{iq_2}e^{i\pi/4}\right) \nonumber\\ &= -\frac{J\chi}{2}e^{-i\pi/4}\left(1 + ie^{-iq_1} + e^{i(q_2-q_1)} + ie^{iq_2}\right) \tag{80}\\ &= -\frac{J\chi}{2}e^{-i\pi/4}e^{-i(q_1-q_2)/2}\left(e^{i(q_1-q_2)/2} + ie^{-(q_1+q_2)/2} + e^{i(q_2-q_1)/2} + ie^{i(q_1+q_2)/2}\right) \tag{81}\\ &= -\frac{J\chi}{2}e^{-i\pi/4}e^{-i(q_1-q_2)/2}\left(2\cos\frac{q_1+q_2}{2} + 2i\cos\frac{q_1-q_2}{2}\right) \tag{82}\end{align}$$

The spectrum is made of energies $\epsilon_{\vec{q}} = \pm J\chi\sqrt{\cos^2\frac{q_1+q_2}{2} + \cos^2\frac{q_1-q_2}{2}}$

At half filling exactly half of the state are occupied, therefore the Fermi energy is at the middle of the spectrum, which is $E_F = 0$ since the spectrum obtained in 4)f. is even.

$E_{\vec{q}} = 0$ implies $(q_1, q_2) \in \{(0,0), (0,\pi), (\pi,0), (\pi,\pi)\}$: the Fermi surface is 0-dimensional, a set of 4 points. The ground states correspond to half-filling due to the constraint $n_{i\uparrow} + n_{i\downarrow} = 1$.