Solution Homework 3

In class, we have seen how superconductors can be efficiently described by an effective macroscopic theory, the Landau-Ginsburg theory. Depending on the ratio κ between the London length λ and the coherence length ξ , the superconductor can remain partially superconducting above the conventional critical magnetic field. In this exercise series, placing ourself in the limit

$$\kappa \gg \frac{1}{\sqrt{2}}, \tag{1}$$

we will study the appearance of the so-called Abrikosov vortices (proposed in 1957) in the superconducting condensate and the order parameter. We remind the expression of the free energy density in the Landau-Ginsburg theory of a superconductor

$$f = f_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m}\left|\left(-i\hbar\vec{\nabla} - \frac{2e\vec{A}}{c}\right)\Psi\right|^2 + \frac{\vec{B}^2}{8\pi}, \tag{2}$$

the Maxwell equation

$$\vec{\mathrm{rot}}\vec{B} = \frac{4\pi}{c}\vec{j} = \frac{4\pi}{c}\left[\frac{-ie\hbar}{m}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) - \frac{4e^2\vec{A}}{mc}\Psi^*\Psi\right] \tag{3}$$

and the last equation that $\vec{A}$ and Ψ must verify to minimise the free energy:

$$\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{2e\vec{A}}{c}\right)^2\Psi = 0. \tag{4}$$

1) To simplify further computations, rewrite the Eqs. (2)-(4) using the following change of variables:

$$\tilde{A} = \frac{A}{H_c\sqrt{2}\lambda}, \qquad \tilde{H} = \frac{H}{H_c\sqrt{2}}, \qquad \tilde{\Psi} = \Psi\frac{\sqrt{\beta}}{\sqrt{-\alpha}}, \qquad \rho = \frac{r}{\lambda} \tag{5}$$

The Eqs. (3)-(4) now only depends on $\tilde{A}$, $\tilde{\Psi}$ and κ.

Due to the change of variable in r, we need to replace $\vec{\nabla}$ by $\lambda^{-1}\vec{\nabla}$ and $\vec{\mathrm{rot}}$ by $\lambda^{-1}\vec{\mathrm{rot}}$. Straightforward substitution leads to

$$f = f_n + \frac{\alpha^2\lambda^3}{\beta}\left(-|\tilde{\Psi}|^2 + \frac{1}{2}|\tilde{\Psi}|^4 + \left|\left(-i\kappa^{-1}\vec{\nabla} - \vec{A}\right)\tilde{\Psi}\right|^2 + \vec{\tilde{B}}^2\right). \tag{6}$$

The coefficient λ^3 comes from the change in variables in the integral

$$F = \iiint f d^3\vec{r} = \iiint f\lambda^3 d^3\vec{\rho}. \tag{7}$$

The remaining two equations are given by

$$\vec{\mathrm{rot}}\vec{\tilde{B}} = \frac{-i}{2\kappa}\left(\tilde{\Psi}^*\vec{\nabla}\tilde{\Psi} - \tilde{\Psi}\vec{\nabla}\tilde{\Psi}^*\right) - \vec{\tilde{A}}\tilde{\Psi}^*\tilde{\Psi}. \tag{8}$$

$$-\tilde{\Psi} + |\tilde{\Psi}|^2\tilde{\Psi} + \left(-i\kappa^{-1}\vec{\nabla} - \vec{\tilde{A}}\right)^2\tilde{\Psi} = 0. \tag{9}$$

2) We now place ourselves at low field (applied along the z direction) and take $\kappa \gg 1$. We look for solutions of the form

$$\tilde{\Psi}(\rho, \theta, z) = u(\rho)e^{i\chi(\theta)}, \text{ with } \chi(\theta) = n\theta \tag{10}$$

What values can n take? What should be $\lim_{\rho \to +\infty} u(\rho)$?

We consider the vector potential in the symmetric gauge and assume (you can show it) $\vec{\tilde{A}} = \tilde{A}(\rho)\vec{e}_\theta$. Rewrite the simplified versions of Eqs. (2)-(4) , defining $v_s = \frac{n}{\kappa\rho} - \tilde{A}(\rho)$. What is the physical meaning of v_s?

The order parameter Ψ must be well-defined, and therefore its phase is quantized, i.e. $n \in \mathbb{Z}$. Far from the vortex, the superconducting density should remain unaffected by its presence. We can therefore take $\lim_{\rho \to +\infty} u(\rho) = 1$.

Due to the form chosen for $\tilde{\Psi}$ and $\tilde{A}$

$$\left(-i\kappa^{-1}\vec{\nabla} - \vec{\tilde{A}}\right)\tilde{\Psi} = -i\kappa^{-1}f'(\rho)e^{in\theta}\vec{e}_\rho + \left(\frac{n}{\kappa\rho} - A(\rho)\right)\tilde{\Psi}\vec{e}_\theta = -i\kappa^{-1}u'(\rho)e^{in\theta}\vec{e}_\rho + v_s\tilde{\Psi}\vec{e}_\theta \tag{11}$$

Similarly,

$$\left(-i\kappa^{-1}\vec{\nabla} - \vec{\tilde{A}}\right)^2\tilde{\Psi} = -\frac{\Delta}{\kappa^2}\tilde{\Psi} + 2i\frac{\vec{\tilde{A}}}{\kappa}\vec{\nabla}\tilde{\Psi} + \vec{\tilde{A}}^2\tilde{\Psi} \tag{12}$$

$$= e^{in\theta}\left(-\frac{1}{\kappa^2\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}u) + u\frac{n^2}{\kappa^2\rho^2} - 2u\frac{n\tilde{A}}{\rho\kappa} + u\tilde{A}^2\right) \tag{13}$$

$$= e^{in\theta}\left(-\frac{1}{\kappa^2\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}u) + v_s^2 u\right) \tag{14}$$

Starting from Eq. (9), *we then obtain:*

$$-u + u^3 + -\frac{1}{\kappa^2\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}u) + v_s^2 u = 0. \tag{15}$$

Similarly, Eq. eq8 transforms into

$$r\vec{o}t\vec{\tilde{B}} = \tilde{j} = v_s u^2. \tag{16}$$

v_s therefore acts as the effective speed of the superconducting condensate.

Finally, the free energy density can be written as:

$$f = f_n + \frac{\alpha^2\lambda^3}{\beta}\left(-u^2 + \frac{1}{2}u^4 + \frac{u'^2}{\kappa^2} + v_s^2u^2 + \tilde{B}^2\right) \tag{17}$$

3) We now place ourselves in the limit $\rho \gg \kappa^{-1}$. Justify that we can approximate $u(\rho)$ by its value at $+\infty$. Using this approximation, show that v_s verifies

$$\rho^2\frac{d^2}{d\rho^2}v_s + \rho\frac{d}{d\rho}v_s - (1 + \rho^2)v_s = 0 \tag{18}$$

From the class, we know that λ is the penetration length of the magnetic field within the superconductor, i.e. the characteristic length of variation of the magnetic field. Similarly, ξ is the coherence length of the superconductor, i.e., the typical length scale upon which the superconducting field vanishes at an interface with normal metal or the vacuum.
The limit $\rho \gg \kappa^{-1}$ corresponds to $r \gg \xi$: the fraction of the condensate should indeed be well stabilized

at this range.

Now let's try to see that from the equations. At the length scales we consider, v_s is dominated by the behavior of A, and therefore $\tilde{B}$ over scales of order 1 (assuming u^2 not too far from 1). On the other hand, u varies on scales $\rho\kappa = \rho/\xi$, and therefore can be assumed to have saturated.

To obtain a differential equation on v_s, we first write Eq. 16 as

$$-\frac{d}{d\rho}\tilde{B}_z = v_s u^2 \tag{19}$$

All the other components of $\tilde{B}$ vanish due to the form of $\tilde{A}$. Using the definition of v_s, we also have

$$\vec{\tilde{A}} = \kappa^{-1}\vec{\nabla}\chi - v_s e_\theta \tag{20}$$

$$\vec{\tilde{B}} = \kappa^{-1}\vec{\mathrm{rot}}\vec{\nabla}\chi - \vec{\mathrm{rot}} v_s \vec{e}_\theta \tag{21}$$

Note that here, the rotational of $\vec{\nabla}\chi$ in principle requires special care. For an arbitrarily small ρ,

$$\oint_{C_\rho} \vec{\nabla}\chi . d\vec{l} = 2\pi n. \tag{22}$$

On the other hand,

$$\oint_{C_\rho} \vec{\nabla}\chi . d\vec{l} = \iint_{D_\rho} \vec{\mathrm{rot}}\vec{\nabla}\chi . d\vec{S}. \tag{23}$$

Here we noted C_ρ and D_ρ the circle and the disk of radius ρ centered in 0. This means that

$$\vec{\mathrm{rot}}\vec{\nabla}\chi = 2\pi n \delta(\rho)\vec{e}_z. \tag{24}$$

If we apply naively the rotational to v_s, we obtain

$$\tilde{B}_z = 2\pi n \kappa^{-1}\delta(\rho) - \frac{1}{\rho}\frac{d}{d\rho}\rho v_s \tag{25}$$

This expression is problematic: the magnetic field has a dirac-like divergence in 0. To avoid this, we require that, similarly to $\vec{\nabla}\chi$, at small ρ

$$\vec{\mathrm{rot}} v_s e_\theta = \left(\frac{1}{\rho}\frac{d}{d\rho}\rho v_s + 2\pi n \kappa^{-1}\delta(\rho)\right)\vec{e}_z. \tag{26}$$

For this to be valid, we will need to enforce

$$v_s \sim \frac{n}{\kappa\rho} \text{ when } \rho \to 0. \tag{27}$$

Taking into account this condition, we finally obtain

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho v_s - v_s u^2 = 0. \tag{28}$$

Identifying $u^2 = 1$ and developing the derivatives, we obtain the equation in the question.

4) The solutions of the previous differential equation are linear combinations of the modified Bessel functions I_1 and K_1. Justify that

$$v_s = n\frac{K_1(\rho)}{\kappa}. \tag{29}$$

What is the asymptotic form of v_s for $\rho \gg 1$? For $\rho \ll 1$? (use reference books, functions.wolfram.com or wikipedia for special functions)

Opening our favourite compendium of special functions (functions.wolfram.com), we note that while I_1 and K_1 are both solutions of the same differential equation, they have different limit behaviours:

- *I_1 is exponentially growing at ∞ and $I_1(0) = 0$*
- *K_1 is exponentially decaying at ∞ and $K_1(\rho) \sim \frac{1}{\rho}$ for $\rho \to 0$*

Both functions have divergences, but at infinity, v_s must vanish as B must be finite, and v_s must diverge as ρ^{-1} at $\rho = 0$. We therefore take

$$v_s = cK_1(\rho). \tag{30}$$

For small ρ, the field is regular (and therefore A), and therefore v_s must diverge as $\kappa^{-1}\vec{\nabla}\xi = \frac{n}{\kappa\rho}$. We finally obtain

$$v_s = \frac{nK_1(\rho)}{\kappa} \tag{31}$$

The asymptotic expansion of v_s at $+\infty$ is given by:

$$v_s \sim \frac{n}{\kappa}\sqrt{\frac{\pi}{2\rho}}e^{-\rho}. \tag{32}$$

v_s therefore decays much faster for $\rho \geq 1$, which we will use as a cut-off length.

5) Using the asymptotic form of v_s, show that $u(\rho) = 1 - \frac{n^2}{2\kappa^2\rho^2} + o((\kappa\rho)^{-4})$ for $\kappa^{-1} \ll \rho \ll 1$.

We can write

$$u = \sum_{m=0}^{2} \frac{u_m}{(\kappa\rho)^m} + o((\kappa\rho)^{-2}) \textit{ with } u_0 = 1. \tag{33}$$

as $\lim_{\rho\to+\infty} u = 1$. We now rewrite the equation

$$-u + u^3 - \frac{1}{\kappa^2\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}u) + v_s^2 u = 0. \tag{34}$$

into

$$\frac{1}{\kappa^2}u'' + \frac{1}{\kappa^2\rho}u' = u^3 - (1 - v_s^2)u. \tag{35}$$

and replace v_s by its asymptotic value

$$v_s = \frac{n}{\kappa\rho} + \delta v_s \textit{ with } \delta v_s = o((\kappa\rho)^{-1}). \tag{36}$$

leading to

$$\frac{1}{\kappa^2}\sum_{m=0}^{2}\frac{m(m+1)u_m}{\kappa^m\rho^{m+2}} - \frac{1}{\kappa^2\rho}\sum_{m=0}^{2}\frac{mu_m}{\kappa^m\rho^{m+1}} = \sum_{m_1+m_2+m_3=m}\frac{u_{m_1}u_{m_2}u_{m_3}}{\kappa^m\rho^m} - (1 - \frac{n^2}{\kappa^2\rho^2})\sum_{m=0}^{+\infty}\frac{u_m}{\kappa^m\rho^m} + o((\kappa\rho)^{-2}). \tag{37}$$

We only kept the larger $-o((\kappa\rho)^{-n}$ term for reference. The equality must be true order by order.

$$0+0 = 1-1 \text{ at order } 0 \tag{38}$$
$$0+0 = 3u_0^2 u_1 - u_1 \text{ at order } 1 \tag{39}$$
$$0-0 = 3u_0^2 u_2 + 3u_0 u_1^2 - (u_2 - u_0 n^2) \text{ at order } 2 \tag{40}$$

The solutions are given by

$$u_1 = 0 \text{ and } u_2 = -\frac{1}{2}n^2. \tag{41}$$

Note: to be formally correct, we had to write u only including terms of order up to 2 in $\kappa\rho$ as $\delta v_s \sim \ln\rho$. In principle, to get the next order, ln *terms should be included. This logarithmic divergence of the v_s field (and therefore of $\tilde{A}$) is not physical, and a consequence of our simplifications: we did not take into account the vanishing density of superconducting current at $\rho = 0$.*

6) Assuming $u \ll 1$ when $\rho \ll \kappa^{-1}$, compute the approximate form of u close to $\rho = 0$ (note: the multiplicative constant cannot be simply obtained).

In that limit, we can neglect the non-linear term u^3. The differential equation becomes:

$$\frac{1}{\kappa^2}u'' + \frac{1}{\kappa^2\rho}u' + (1 - \frac{n^2}{\kappa^2\rho^2})u = 0 \tag{42}$$

In the limit $\rho \ll \kappa^{-1}$, we can further approximate $(1 - \frac{n^2}{\kappa^2\rho^2})$ by $-\frac{n^2}{\kappa^2\rho^2}$. The differential equation then admits polynomial solutions

$$u = C\rho^n + D\rho^{-n}. \tag{43}$$

As we assumed $u \ll 1$, we have

$$f = C\rho^n = d\kappa^n\rho^n. \tag{44}$$

The numerical constant d can only be obtained through numerical integration.

7) Represent the profile of v_s, u and $\tilde{B}$ on the same graphic. How much flux is carried by a vortex?

The schematic behavior of v_s, u and $\tilde{B}$ can be found in Fig. 1. For $\tilde{B}$, it is more convenient to work with the equation

$$\frac{dB_z}{d\rho} = -u^2 v_s \tag{45}$$

to get the correct long range scaling. For $\rho \gg 1$, it is exponentially suppressed due to K_1. For $\kappa^{-1} \ll \rho \ll 1$, it grows logarithmically up to

$$\tilde{B}_z \approx \frac{\ln\kappa}{\kappa} \text{ for } \rho = \kappa^{-1} \tag{46}$$

Finally, close to 0, we have

$$\tilde{B}_z \approx \frac{d^2\kappa^{2n-1}}{2n}(\kappa^{-2n} - \rho^{-2n}) + \tilde{B}_z(\kappa^{-1}) \sim \frac{\ln\kappa}{\kappa} \tag{47}$$

$\tilde{B}_z$ is indeed quasi-constant within the core of the vortex.

The speed v_s vanishes for $\rho \gg 1$. The potential vector therefore then exactly compensate $\vec{\nabla}\chi = \frac{n}{\kappa\rho}$. The magnetic flux generated by the vortex is consequently

$$\tilde{\Phi} = \iint \vec{\tilde{B}}.d\vec{S} = \int_{C_\infty} \vec{\tilde{A}}.d\vec{l} = \frac{2\pi n}{\kappa}. \tag{48}$$

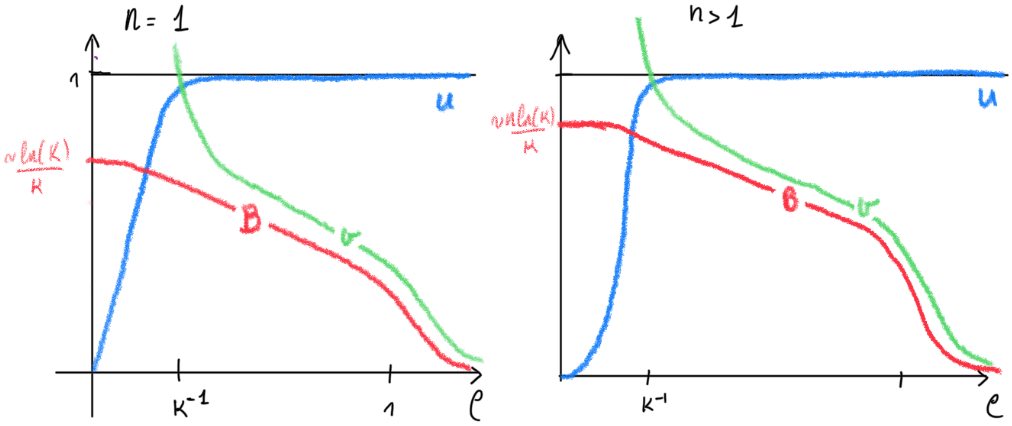

Figure 1: Representation of v_s, u and $\tilde{B}$ for $n = 1$ and $n > 1$.

Each vortex therefore carries n elementary flux $\frac{2\pi}{\kappa}$. Rewriting this result in conventional units, we obtain the standard quantization of the magnetic flux in a superconductor.

Now that we understand the basic properties of vortices, we try to estimate the first critical field of a superconductor $H_{c,1}$, i.e., the field at which the first vortex appears.

8) Justifying why we can neglect the other regions , compute the leading contribution to the kinetic energy associated to the presence of a single vortex for $\kappa^{-1} \ll \rho \ll 1$.

We recall the expression of the free energy density obtained in question 2)

$$f = f_n + \frac{\alpha^2 \lambda^3}{\beta} \left(-u^2 + \frac{1}{2}u^4 + \frac{u'^2}{\kappa^2} + v_s^2 u^2 + \tilde{B}^2 \right) \tag{49}$$

Removing the potential and the magnetic terms, the kinetic energy density is given by:

$$\frac{\alpha^2 \lambda^3}{\beta} \left(\frac{u'^2}{\kappa^2} + v_s^2 u^2 \right) \tag{50}$$

For $\rho \gg 1$, u is constant and v_s is exponentially suppressed and can be neglected, therefore the integral domain can be limited to $\rho < \kappa^{-1}$ (this means $r < \xi$: we neglect anything beyond one correlation length, acceptable). For $\rho \ll \kappa^{-1}$, the diverging v_s is compensated by the vanishing density of Cooper pair. The kinetic energy in that region can be approximated by

$$E^n_{\text{kin},1} = 2\pi L_z \lambda^2 \frac{\alpha^2}{\beta} \int_0^{\kappa^{-1}} \rho d\rho \left(d^2 n^2 \kappa^{2n-2} \rho^{2n-2} + d^2 \kappa^{2n} \rho^{2n} \times \frac{n^2}{\kappa^2 \rho^2} \right) \tag{51}$$

$$= 2\pi L_z \lambda^2 \frac{\alpha^2}{\beta} n \frac{\kappa^{2n-2}}{\kappa^{2n}} = \frac{2\pi \lambda^2 L_z \alpha^2 n}{\beta \kappa^2} \tag{52}$$

The main contribution to the free energy and the kinetic energy therefore arises from $\kappa^{-1} \ll \rho \ll 1$ (note that none of the bounds are exact, we will only get the scaling of the kinetic energy). Replacing

v_s and u by their approximation, the leading contribution is given by the term in $v_s^2u^2$ such that

$$E_{\rm kin}^n \approx 2\pi\lambda^2 L_z \frac{\alpha^2}{\beta} \int_{\kappa^{-1}}^{1} \rho d\rho \frac{n^2}{\kappa^2\rho^2} \tag{53}$$

$$= 2\pi\lambda^2 L_z \frac{\alpha^2}{\beta} n^2 \frac{\ln\kappa}{\kappa^2}. \tag{54}$$

9) Due to the presence of the superconducting currents, the line vortices have a finite magnetic moment and interact with external magnetic fields. In the presence of an external magnetic field, the proper free energy to consider is the Gibbs free energy which is here simply given by

$$f_G = f - \frac{\vec{B}.\vec{H}}{4\pi}. \tag{55}$$

The additional term is simply the work generated by the external magnetic field applied on the vortex line. Find the first critical magnetic field by comparing the free energy in the presence of a vortex with the free energy of a superconductor without vortices.

Vortices appear when the free energy in the presence of a vortex becomes smaller than the free energy of a uniform superconductor. The critical magnetic field verifies

$$f_G^{n=1} = f_G^{n=0}. \tag{56}$$

The magnetic energy induced by the current can be neglected in front of the kinetic energy: indeed, we know that $\tilde{B}$ varies over a scale $\rho = 1$ and with a total flux $\frac{2\pi}{\kappa}$, is suppressed at large ρ. A first approximation for $\tilde{B}_z$ is therefore $\tilde{B}_z \approx \frac{2}{\kappa}$. This leads to a magnetic energy of order κ^{-2}. We therefore obtain

$$E_{\rm kin} = 2\lambda^2 L_z \frac{\alpha^2}{\beta} \int d^2\vec{\rho}\vec{\tilde{H}}.\vec{\tilde{B}}. \tag{57}$$

The total magnetic flux generated by the Cooper pairs condensate is given by

$$\tilde{\Phi} = \int \vec{\tilde{A}}.d\vec{l} = \frac{2\pi n}{\kappa}. \tag{58}$$

The difference in free energy between a superconductor with one vortex and a superconductor without vortex is therefore given by

$$\Delta f = L_z \frac{\alpha^2\lambda^2}{\beta}\left(2\pi n^2 \frac{\ln\kappa}{\kappa^2} - \tilde{H}\frac{4\pi n}{\kappa}\right). \tag{59}$$

This is minimized for $n = 1$ and the critical temperature above which vortices appear is

$$\tilde{H}_{c,1} = \frac{\ln\kappa}{2\kappa}. \tag{60}$$

In normal units,

$$H_{c,1} = \frac{H_c \ln\kappa}{\sqrt{2}\kappa}. \tag{61}$$

Vortices start to appear at much lower magnetic fields than H_c.

Above the critical field $H_{c,1}$, the superconducting condensate can therefore present vortices. The rest of the exercise is devoted to the study of their physics.

10) Using simple energetic arguments, justify that it is more advantageous to create two vortices with $n = 1$ than one vortex with $n = 2$.

As seen in the previous two questions, a vortex has a kinetic energy proportional to n^2. It is therefore more favorable to split the vortex into n vortices with $n = 1$, supposing that they are arbitrarily far from each other and do not interact.

11) Consider two vortices with $n = 1$, centered in $(0, 0)$ and $(a, 0)$, with $\kappa^{-1} \ll a \ll 1$. We assume that v_s can be well approximated by the sum of the contributions of each vortices far from their core. Computing the difference between their kinetic energy and the kinetic energy of two holes with $n = 1$, estimate the interaction between the vortices. Is the interaction repulsive or attractive ?

Hint: once you have justified that the main region of interest is situated within 1 and at more than κ^{-1} of each vortex, compute the kinetic energy over the following region:

$$I = \{\vec{r} \in \mathbb{R}^2 | \min(\rho, |\vec{\rho} - a\vec{e}_x|) < 1\}, \tag{62}$$

using the asymptotic form for the region $\rho \ll 1$ for both vortex contributions. While we are slightly deforming the proper region, it will give us the right dominant contribution. We give the formula

$$\int dr \frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} = \ln\left(r - a\cos\theta + \sqrt{r^2 + a^2 - 2ar\cos\theta}\right) \tag{63}$$

The previous question already qualitatively answers the question: if it is energetically unfavourable to have two vortices fused at the same point, vortices must repulse each other, at least at short distance. We have $v_s \approx \frac{1}{\kappa}(K_1(\rho) + K_1(|\vec{\rho} - a\vec{e}_x|)$ The kinetic energy of the pair of vortex is dominated by

$$E_{\text{kin}} = L_z\lambda^2 \frac{|\alpha|^2}{\beta} \iint d^2\vec{\rho} v_s^2 u(\rho)^2 \tag{64}$$

For the same reasons as in the previous questions, we can focus on the region where ρ is at more than κ^{-1} than each vortex, and not too far from any due to the exponential cut-off. There, we can approximate $K_1(\rho)$ by $\frac{1}{\kappa\rho}$ and u by 1 leading to

$$\begin{aligned}
E_{\text{kin}} &= 2E_{\text{kin}}^{n=1} + 2L_z\lambda^2 \frac{|\alpha|^2}{\kappa^2\beta} \iint d^2\vec{\rho} \frac{1}{\rho} \frac{1}{\sqrt{\rho^2 + a^2 - 2a\rho\cos\theta}} \\
&= 2E_{\text{kin}}^{n=2} + 4L_z\lambda^2 \frac{|\alpha|^2}{\kappa^2\beta} \iint_I d^2\vec{\rho} \frac{1}{\rho} \frac{1}{\sqrt{\rho^2 + a^2 - 2a\rho\cos\theta}}
\end{aligned}$$

where the region I is defined in Fig. 2. Note that this domain is chosen for convenience in the integration: we select all $\vec{\rho}$ that are less than 1 of at least one vortex. This partially changes the integral, does not affect the dominant scaling, but significantly simplifies the computation (that is already challenging).

We then have

$$\begin{aligned}
\Delta E_{\text{kin}} &= 4L_z\lambda^2 \frac{|\alpha|^2}{\kappa^2\beta} \int_0^{2\pi} d\theta \int_{r_-(\theta)}^{r_+(\theta)} d\rho \frac{1}{\sqrt{\rho^2 + a^2 - 2a\rho\cos\theta}} \\
&= 4L_z\lambda^2 \frac{|\alpha|^2}{\kappa^2\beta} \int_0^{2\pi} d\theta \left[\ln\left(\rho - a\cos\theta + \sqrt{a^2 + \rho^2 - 2a\rho\cos\theta}\right)\right]_{r_-(\theta)}^{r_+(\theta)} \\
&= 4L_z\lambda^2 \frac{|\alpha|^2}{\kappa^2\beta} (I_+ - I_-) .
\end{aligned}$$

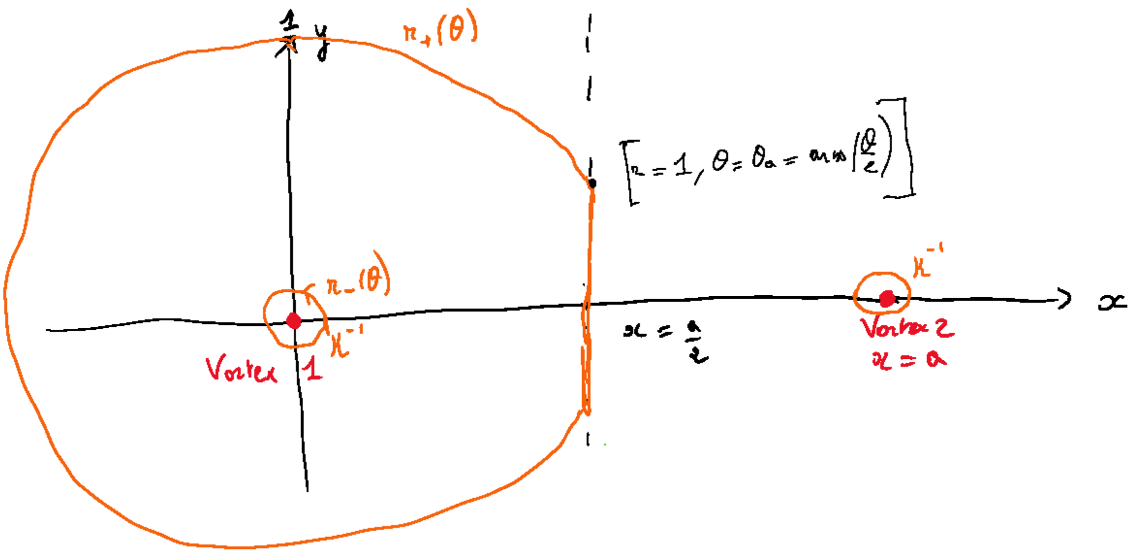

Figure 2: Domain of integration for the question 11.

We have $r_-(\theta) = \kappa^{-1} \ll a$: the integral I_- can therefore be approximated by $2\pi \ln(a)$. The integral on the second contour is significantly trickier, but less relevant. $r_+(\theta) = \frac{a}{2\cos\theta}$ for $\theta \in [-\arccos(\frac{a}{2}), \arccos(\frac{a}{2})] = [-\theta_a, \theta_a]$, 1 else. We can then write the integral on the second contour as:

$$I_+ = 2\int_0^{\theta_a} d\theta \ln a \frac{1-\cos^2\theta}{\cos\theta} + 2\int_{\theta_a}^{\pi} d\theta \ln\left(\sqrt{a^2+1-2a\cos\theta} - a\cos\theta + 1\right)$$

Due to the presence of the bounds, we need to be a bit careful here. The simplest method is to evaluate the derivative of I_+ with respect to a. By continuity, we realize that the boundary terms do not contribute, and we are left with

$$\partial_a I_+ = \frac{2\theta_a}{a} + 2\int_{\theta_a}^{\pi} \frac{d\theta_a}{a}\left(1 - \frac{1}{\sqrt{1+a^2-2a\cos\theta}}\right) \tag{65}$$

We now use the fact that $a \ll 1$ and develop the square root (note that the integral can be performed exactly but involves elliptic functions so we will skip it). At leading order in a, we obtain

$$2\int_{\theta_a}^{\pi} \cos\theta = -2\sqrt{1-\frac{a^2}{4}} \tag{66}$$

Given that $a \ll 1$, I_+, we can neglect the a dependency of this term compared to the previous logarithmic contribution.

$$I_+ \approx 2\int da\frac{\theta_a}{a} \approx 2\int da \frac{\pi}{2a} \approx \pi \ln a \tag{67}$$

Finally, we obtain that

$$E_{\text{kin}} = 2E_{\text{kin}}^{n=1} - 4L_z\lambda^2\frac{|\alpha|^2}{\kappa^2\beta} \times \pi\ln(a) = 2E_{\text{kin}}^{n=1}\left(1 - \frac{\ln a}{\ln\kappa}\right) = E_{\text{kin}}^{n=2} - 2E_{\text{kin}}^{n=1}\ln(a) \tag{68}$$

The two vortices therefore indeed repulse each other (the energy decreases when a increases), and the interaction between two vortices decrease logarithmically with the distance between them. When $a \approx 1$, i.e., the holes are separated by approximately λ, they cease repulsing each other (within our approximation).

12) In the absence of any other factor, what do you expect the vortices to do at small magnetic field?

At small magnetic field, above $H_{c,1}$, a small number of vortices are present in the superconductor. They can move around freely while remaining at a distance larger than λ from each other. In practice, due to the small residual interaction, if the superconductor is too small, the vortices may push each other out of the superconductor, leading to a permanent circulation of vortices in the system. Vortices form close to the middle of the sample and push each other to the edges where they are destroyed before reforming again. This phenomenon can also be observed in conventional superfluids in rotation.

13) In fact, vortices tend to be pinned by impurities and form a lattice. Can you intuit the structure of the network they form at high field? Estimate from that $H_{c,2}$ as a function of κ, neglecting the kinetic energy of the vortices.

In a densely packed lattice of vortices, the superconducting density is close to 0 everywhere. At $H_{c,2}$, we should then have

$$f_G = f_n, \tag{69}$$

and therefore, neglecting the kinetic energy, we obtain

$$\iint \vec{\tilde{B}}^2 - 2\vec{\tilde{H}}\vec{\tilde{B}} = 0. \tag{70}$$

Taking $\tilde{B}$ to be approximately constant (an approximation valid if the distant between vortices is much smaller than 1), we obtain

$$\vec{\tilde{H}} = \frac{1}{2}\vec{\tilde{B}}. \tag{71}$$

Each vortex carries a flux $\frac{2\pi}{\kappa}$, leading to a local magnetic field $\tilde{B}_z = \frac{2\pi}{S_1\kappa}$, where S_1 is the area occupied by a single vortex (i.e., S_1^{-1} is the density of vortex line). To maximise the number of vortices and therefore the flux, the vortices organize themselves into the densest possible networt, that is to say a triangular lattice. A unit-cell occupies an area $\frac{a^2\sqrt{3}}{4}$ where a is the distance between two vortices. At $H_{c,2}$, the distance between two vortices should be of order κ^{-1}, leading to

$$\tilde{H}_{c,2} \approx \frac{4\pi}{\sqrt{3}}\kappa \tag{72}$$

$$H_{c,2} \approx \frac{4\pi\sqrt{2}}{\sqrt{3}}H_c\kappa \tag{73}$$

While the coefficient is more than approximate, we capture the behaviour in κ of the critical field (compare with the result obtained in class). The larger κ, the larger the range of magnetic field where vortices are present in the superconductor. A more precise computation of the second critical magnetic field was performed by Abrikosov in 1958. It takes into account the kinetic energy of the vortices, which lead to a more convoluted analysis. The starting point is to consider a superposition of solutions of the Landau level equation. In fact, amusingly, Abrikosov made a small mistake in this very technical computation and initially predicted a square lattice of vortices.