

This homework aims to introduce some concepts of topology by comparing a minimal model of Chern insulator and the quantum Hall effect. The Chern number, which we will introduce below, is a first example of what we call a topological invariant. It is a quantity which is not sensitive to small, smooth, local deformations of the system we study. Its geometric interpretation is the following: considering a function f mapping the sphere onto itself, (or a torus onto the sphere), it counts the number of times our function f will cover the sphere. As a small example: $f(\vec{r}) = (1, 0)$ has a Chern of 0, $f(\vec{r}) = \vec{r}$ has Chern 1 and $f(\vec{r}) = -\vec{r}$ has Chern -1 . We will show this Chern number appears naturally in the derivation of the Hall conductance.

While seemingly long, the homework can be done relatively quickly. Part 1 involves the most computations, using basic concepts of second quantization. Use the provided hints to minimize calculations. Some computations should be done numerically as they do not admit simple close forms (or they are beyond what I expect you to be able to do). In that case, you can use your software of choice. Free softwares include python (with numpy + scipy + matplotlib) or Julia (with HCubature.jl + PythonPlot.jl/Plots.jl) for example. The last section (relating current and Chern) is trickier: we will not try to derive the Kubo formula. Part 2 is actually relatively computation free, if you follow the hints.

Part 1: A minimal Chern insulator

Consider the following two-band model on a square lattice (the Qi-Wu-Zhang model):

$$H = \sum_{\vec{k} \in \text{BZ}} \begin{pmatrix} c_{\vec{k}, \uparrow}^\dagger & c_{\vec{k}, \downarrow}^\dagger \end{pmatrix} h(\vec{k}) \begin{pmatrix} c_{\vec{k}, \uparrow} \\ c_{\vec{k}, \downarrow} \end{pmatrix} \quad (1)$$

$$h(\vec{k}) = \sin(k_x)\sigma_x + \sin(k_y)\sigma_y + [m - \cos(k_x) - \cos(k_y)]\sigma_z \quad (2)$$

where σ_i are Pauli matrices, and m is a tunable mass parameter. In general, we will work at half-filling.

Band structure and phase diagram

1. Express the model in real space. Comment on the nature of the different hopping terms. Do you think it is very realistic?

Correction: We simply want here to perform an inverse Fourier transform. We remind:

$$c_{\vec{k}} = \frac{1}{N} \sum_{\vec{r}} e^{-i\vec{k} \cdot \vec{r}} c_{\vec{r}}.$$

From there,

$$\begin{aligned} \sum_{\vec{k}} c_{\vec{k}, \alpha}^\dagger c_{\vec{k}, \beta} &= \sum_{\vec{r}} c_{\vec{r}, \alpha}^\dagger c_{\vec{r}, \beta} \\ \sum_{\vec{k}} \cos k_x c_{\vec{k}, \alpha}^\dagger c_{\vec{k}, \beta} &= \frac{1}{2} \sum_{\vec{r}} c_{\vec{r}, \alpha}^\dagger c_{\vec{r} + \vec{e}_x, \beta} + c_{\vec{r} + \vec{e}_x, \alpha}^\dagger c_{\vec{r}, \beta} \\ \sum_{\vec{k}} \sin k_x c_{\vec{k}, \alpha}^\dagger c_{\vec{k}, \beta} &= \frac{1}{2i} \sum_{\vec{r}} c_{\vec{r}, \alpha}^\dagger c_{\vec{r} + \vec{e}_x, \beta} - c_{\vec{r} + \vec{e}_x, \alpha}^\dagger c_{\vec{r}, \beta} \end{aligned}$$

Putting everything together, and denoting $\Psi^\dagger(\vec{r}) = \begin{pmatrix} c_{\vec{r},\uparrow}^\dagger & c_{\vec{r},\downarrow}^\dagger \end{pmatrix}$

$$H = \sum_{\vec{r}} m \Psi^\dagger(\vec{r}) \sigma^z \Psi(\vec{r}) - \Psi^\dagger(\vec{r}) \left(\frac{\sigma^z}{2} - \frac{\sigma^x}{2i} \right) \Psi(\vec{r} + \vec{e}_x) - \Psi^\dagger(\vec{r} + \vec{e}_x) \left(\frac{\sigma^z}{2} + \frac{\sigma^x}{2i} \right) \Psi(\vec{r}) \\ - \Psi^\dagger(\vec{r}) \left(\frac{\sigma^z}{2} - \frac{\sigma^y}{2i} \right) \Psi(\vec{r} + \vec{e}_y) - \Psi^\dagger(\vec{r} + \vec{e}_y) \left(\frac{\sigma^z}{2} + \frac{\sigma^y}{2i} \right) \Psi(\vec{r}).$$

This Hamiltonian includes a Zeeman field m , and only spin-orbit couplings for hoppings. As such, it is generally quite unphysical (except as an effective model for atoms deposited on top of a magnetic surface).

2. Calculate the energy spectrum $E_\pm(\vec{k})$. For what values of m does the system become gapless?

Correction: We recall that the spectrum of the Hamiltonian $H = \vec{n} \cdot \vec{\sigma}$ is simply $E_\pm = \pm |\vec{n}|$. Consequently,

$$E_\pm(\vec{k}) = \pm \sqrt{m^2 + 2 - 2m \cos k_x - 2m \cos k_y + 2 \cos k_x \cos k_y}.$$

The system becomes gapless when $E_+(\vec{k}) = E_-(\vec{k})$, i.e. when $|\vec{n}| = 0$. Cancelling the x and y component requires $k_{x/y} = 0 \bmod \pi$. The gap closes then for $m = 2$ ($\vec{k} = (0, 0)$), for $m = -2$ ($\vec{k} = (\pi, \pi)$) and for $m = 0$ ($\vec{k} = (\pi, 0)$ or $(0, \pi)$).

3. Represent the band structure close to the gap-closing point when the system is gapless. What structure do you have at the gap closing points?

Correction: Close to the gap closing points, the spectrum is linear. For example, for $m = 2$, the gap close at $(0, 0)$. Expanding at first order in \vec{k} , we obtain

$$H \approx k_x \sigma^x + k_y \sigma^y \Rightarrow E_\pm = \pm k.$$

The results are similar, up to some signs, at all gap closing points. The magnetic structure is the structure of a Dirac point, similar to the gap closing at the K and K' points in graphene.

4. Let us define the matrix $M = \begin{pmatrix} \cos \theta & e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & -\cos \theta \end{pmatrix}$. Show that the eigenvectors associated to the eigenvalues ± 1 are $(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2})$ and $(-e^{i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2})$.

Correction: Simply replace and verify the eigenvector equations. I strongly recommend knowing this trick.

5. Express the density $n = \langle n_\uparrow + n_\downarrow \rangle$ and the magnetization $s = \langle n_\uparrow - n_\downarrow \rangle$ in the groundstate, as an explicit integral over \vec{k} .

Correction: We work at half-filling, so by definition, $n = 1$. At half-filling, the groundstate is simply $\prod_{\vec{k}} c_{\vec{k},-}^\dagger |0\rangle$ and

$$s = \frac{1}{N} \sum_{\vec{k}} \langle n_{\vec{k},\uparrow} - n_{\vec{k},\downarrow} \rangle.$$

We define

$$\cos \theta_{\vec{k}} = \frac{m - \cos k_x - \cos k_y}{E_+(\vec{k})} \\ e^{i\phi_{\vec{k}}} \sin \theta_{\vec{k}} = \frac{\sin k_x - i \sin k_y}{E_+(\vec{k})}$$

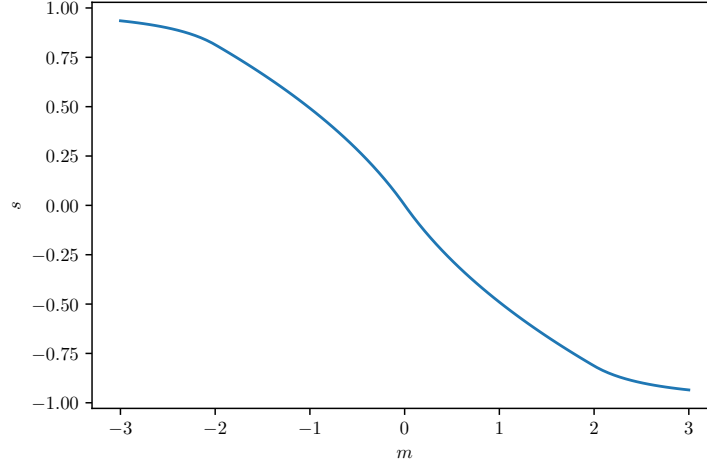


Figure 1: Magnetization (blue) and its derivative (blue) as a function of m .

Using the previous question, we see that

$$c_{\vec{k},\uparrow} = -e^{i\phi} \sin \frac{\theta}{2} c_{\vec{k},-} + \cos \frac{\theta}{2} c_{\vec{k},+}$$

$$c_{\vec{k},\downarrow} = e^{-i\phi} \sin \frac{\theta}{2} c_{\vec{k},+} + \cos \frac{\theta}{2} c_{\vec{k},-}$$

and therefore

$$s = \frac{1}{N} \sum_{\vec{k}} \sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2} = -\frac{1}{N} \sum_{\vec{k}} \cos \theta_{\vec{k}} \approx - \int_{BZ} \frac{d\vec{k}}{4\pi^2} \frac{m - \cos k_x - \cos k_y}{E_+(\vec{k})}$$

6. Compute numerically and plot the magnetization and the density as a function of m . Do you see an order parameter appear? Any singularity?

Correction: The density is constant, so clearly it is not singular. We represent below the magnetization (see Fig. 1). As you can see, it remains continuous throughout the parameter range, and does not act as an order parameter. In fact, its derivative is also continuous, and we only see discontinuities of its second derivative at the transition points.

Berry curvature and Chern number

1. Compute the Berry connection $\vec{A}(\vec{k}) = i\langle u(\vec{k}) | \vec{\nabla}_{\vec{k}} | u(\vec{k}) \rangle$ for the lower band. Note that we want to fix the gauge such that $|u(\vec{k})\rangle$ is periodic over the Brillouin zone. Hint: show that $\vec{A}(\vec{k}) = -\sin^2 \theta / 2 \vec{\nabla}_{\vec{k}} \phi = -\sin^2 \theta / 2 \text{Im} e^{-i\phi} \vec{\nabla}_{\vec{k}} (e^{i\phi})$

Correction: Using the notation of the previous section,

$$\begin{aligned} A_{\alpha} &= i \cos \theta / 2 \times \partial_{\alpha} (\cos \theta / 2) + i \sin \theta / 2 \times \partial_{\alpha} (\sin \theta / 2) - \sin^2 \theta / 2 \partial_{\alpha} \phi \\ &= \frac{i}{2} \partial_{\alpha} (\cos^2 \theta / 2 + \sin^2 \theta / 2) - \sin^2 \theta / 2 \partial_{\alpha} \phi \\ &= -\sin^2 \theta / 2 \partial_{\alpha} \phi \end{aligned}$$

This can be reformulated as

$$A_\alpha = -\sin^2 \theta / 2 \operatorname{Im} (e^{-i\phi} \vec{\nabla}_k e^{i\phi})$$

Now, we can define:

$$\begin{aligned} \sin^2 \theta / 2 &= \frac{1 - \cos \theta}{2} \\ e^{i\phi} &= \frac{\sin k_x - i \sin k_y}{\sqrt{\sin^2 k_x + \sin^2 k_y}} \end{aligned}$$

And we obtain

$$\begin{aligned} A_x &= -\frac{\cos k_x \sin k_y}{\sin^2 k_x + \sin^2 k_y} \times \sin^2 \theta / 2 \\ A_y &= \frac{\sin k_x \cos k_y}{\sin^2 k_x + \sin^2 k_y} \times \sin^2 \theta / 2 \end{aligned}$$

2. Derive the Berry curvature $\Omega(\vec{k}) = \vec{e}_z \cdot (\vec{\nabla}_k \times \vec{A}(\vec{k}))$. Hint: use $\vec{A}(\vec{k}) = -\sin^2 \theta / 2 \vec{\nabla}_k \phi$ to simplify the computation.

Correction:

$$\Omega = \partial_x A_y - \partial_y A_x$$

That means that

$$\Omega = -\partial_x (\sin^2 \theta / 2) \partial_y \phi + \partial_y (\sin^2 \theta / 2) \partial_x \phi = \frac{1}{2} (\partial_x (\cos \theta) \partial_y \phi - \partial_y (\cos \theta) \partial_x \phi)$$

After some straightforward computations, we obtain:

$$\Omega = \frac{m \cos k_x \cos k_y - \cos k_x - \cos k_y}{2 (m^2 - 2m \cos k_y - 2m \cos k_x + 2 \cos k_x \cos k_y + 2)^{3/2}}$$

3. Bonus (very hard): Show that the Chern number $C = \frac{1}{2\pi} \int_{\text{BZ}} \Omega_z(\vec{k}) d^2 k$ takes integer values.

Correction: We will limit ourselves to the case of the torus. What we would like is to use the Stokes theorem

$$C = \frac{1}{2\pi} \int_{\text{BZ}} (\vec{\nabla} \times \vec{A}) \cdot \vec{e}_z d^2 k \quad (3)$$

$$= \oint_{\partial \text{BZ}} \vec{A} \cdot d\vec{l} \quad (4)$$

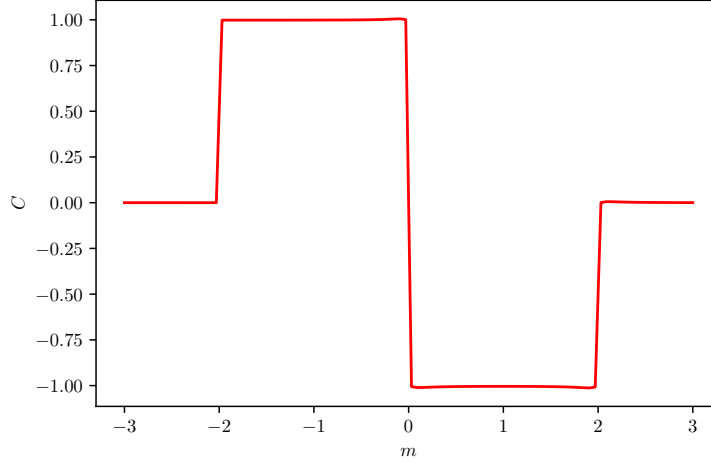
We would naively obtain 0 always. The problem is that the Stokes theorem requires a continuous vector field \vec{A} : it is not always the case. Instead, we can do something slightly more careful. Instead of integrating over the full BZ, we can define the Berry phase

$$\gamma_x(k_y) = \oint_{-\pi}^{\pi} dk_x A_x \quad (5)$$

and similarly for γ_y . Using the Stokes theorem on a ribbon, we obtain that

$$C = \frac{1}{2\pi} \int \partial_y \gamma_x(k_y) dk_y \quad (6)$$

Note that there is a subtlety due to the nature of the torus to avoid double counting γ_x and γ_y . Now, due to the singled-valuedness of the wavefunctions, we know that $\gamma_x(0)$ and $\gamma_x(2\pi)$ should differ by a multiple of 2π . Hence the results.

Figure 2: Chern number as a function of m .

4. Calculate C numerically or analytically (very hard) as a function of m and construct the phase diagram. Phases with different C are said to be topologically distinct.

Correction: Numerically, we find that $C = 0 \forall |m| > 2$, $C = 1 \forall -2 < m < 0$ and $C = -1 \forall 0 < m < 2$. The dependence of the Chern number on m is illustrated in Fig. 2

Transport properties from linear response

We will admit the general Kubo (linear response theory) for the conductivity tensor:

$$\sigma_{\mu\nu}(\omega) = -\frac{1}{\hbar\omega V} \int_0^\infty dt e^{i(\omega+i\eta)t} \langle [\hat{J}_\mu(t), \hat{J}_\nu(0)] \rangle \quad (7)$$

where \hat{J}_μ is the current operator component, and $\eta \rightarrow 0^+$ ensures causality. Here the time evolution under the integral corresponds to the Hamiltonian without electric field.

1. Using the Lehmann representation, expand the expectation value in the energy eigenbasis $\{|n\rangle\}$ of the unperturbed Hamiltonian H .

Correction: We work here in the many-body basis. To simplify notations, we directly assume $T = 0$. We write $J_\mu(t) = e^{iH/\hbar t} J_\mu e^{-iH/\hbar t}$ such that

$$\sigma_{\mu\nu}(\omega) = -\frac{1}{\hbar\omega V} \int_0^\infty dt e^{i(\omega+i\eta)t} \sum_n e^{-i(E_n - E_0)/\hbar t} \langle 0 | \hat{J}_\mu | n \rangle \langle n | \hat{J}_\nu | 0 \rangle - e^{i(E_n - E_0)/\hbar t} \langle 0 | \hat{J}_\nu | n \rangle \langle n | \hat{J}_\mu | 0 \rangle \quad (8)$$

$$\sigma_{\mu\nu}(\omega) = \frac{1}{\omega V} \sum_n \frac{\langle 0 | \hat{J}_\mu | n \rangle \langle n | \hat{J}_\nu | 0 \rangle}{i(\hbar\omega - \Delta E) - \hbar\eta} - \frac{\langle 0 | \hat{J}_\nu | n \rangle \langle n | \hat{J}_\mu | 0 \rangle}{i(\hbar\omega + \Delta E) - \hbar\eta} \quad (9)$$

$$\sigma_{\mu\nu}(\omega) = -\frac{i}{\omega V} \sum_n \frac{\langle 0 | \hat{J}_\mu | n \rangle \langle n | \hat{J}_\nu | 0 \rangle}{\Delta E - \hbar\omega} - \frac{\langle 0 | \hat{J}_\nu | n \rangle \langle n | \hat{J}_\mu | 0 \rangle}{\hbar\omega + \Delta E} \quad (10)$$

In the limit ω goes to 0, getting rid of the diverging terms (they cancel) and expanding the fractions to second-order, we obtain

$$\sigma_{\mu\nu}(\omega) = \frac{i\hbar}{V} \sum_n \frac{\langle 0 | \hat{J}_\mu | n \rangle \langle n | \hat{J}_\nu | 0 \rangle}{\Delta E^2} - \frac{\langle 0 | \hat{J}_\nu | n \rangle \langle n | \hat{J}_\mu | 0 \rangle}{\Delta E^2}. \quad (11)$$

2. In the limit $\omega \rightarrow 0$, we can replace the current operator by $-e\hat{v}_\mu$. Justify that in second quantization, this operator takes the form

$$\hat{J}_\mu = -e \sum_{m,n,\vec{k}} \langle m, \vec{k} | \hat{v}_\mu | n, \vec{k} \rangle c_{m,\vec{k}}^\dagger c_{n,\vec{k}} \quad (12)$$

where m, n are band indices, and $\hat{v}_\mu = \frac{i}{\hbar} [H_0, \hat{r}_\mu]$ is the velocity operator.

Correction: Follow straightforwardly from the class on second quantization, as the velocity is a momentum conserving operator.

3. Show that the commutator $[\hat{J}_x(t), \hat{J}_y(0)]$ creates electron-hole pair excitations. Justify that for a gapped system at $T = 0$, the only non-zero matrix elements are between the ground state and states with single electron-hole pairs.

Correction: Straightforward from the previous expression at $T = 0$

4. Substitute these results into the Kubo formula and take the limit $\omega \rightarrow 0$. You should obtain:

$$\sigma_{xy} = \frac{ie^2\hbar}{V} \sum_{m,n,\vec{k}} \frac{\langle m, \vec{k} | \hat{v}_x | n, \vec{k} \rangle \langle n, \vec{k} | \hat{v}_y | m, \vec{k} \rangle - \langle m, \vec{k} | \hat{v}_y | n, \vec{k} \rangle \langle n, \vec{k} | \hat{v}_x | m, \vec{k} \rangle}{(E_m(\vec{k}) - E_n(\vec{k}))^2}, \quad (13)$$

with V the volume of the system.

Correction: We just need to substitute the expression obtained in 2). We obtain

$$\sigma_{xy} = \frac{ie^2\hbar}{V} \sum_{m \in \text{occupied}, n \in \text{empty}, \vec{k}} \frac{\langle m, \vec{k} | \hat{v}_x | n, \vec{k} \rangle \langle n, \vec{k} | \hat{v}_y | m, \vec{k} \rangle - \langle m, \vec{k} | \hat{v}_y | n, \vec{k} \rangle \langle n, \vec{k} | \hat{v}_x | m, \vec{k} \rangle}{(E_m(\vec{k}) - E_n(\vec{k}))^2}, \quad (14)$$

5. Using this formula, in the continuum limit, justify that

$$|\sigma_{xy}| = \frac{e^2}{h} |C|. \quad (15)$$

Hint: you can use the relation, valid at momentum k , that $v_\mu \sim \frac{\partial_{k_\mu} H(k)}{\hbar}$.

Correction: If we use the definition of the velocity operators given above, we need to take into account properly the boundaries. Instead

$$\sigma_{xy} = \frac{ie^2\hbar}{V} \sum_{m \in \text{occupied}, n \in \text{empty}, \vec{k}} \frac{\langle m, \vec{k} | \hat{v}_x | n, \vec{k} \rangle \langle n, \vec{k} | \hat{v}_y | m, \vec{k} \rangle - \langle m, \vec{k} | \hat{v}_y | n, \vec{k} \rangle \langle n, \vec{k} | \hat{v}_x | m, \vec{k} \rangle}{(E_m(\vec{k}) - E_n(\vec{k}))^2}, \quad (16)$$

Part 2: Landau levels and Chern numbers

Consider a 2D electron gas in a perpendicular magnetic field B described by the Hamiltonian:

$$H = \frac{1}{2m} (\vec{p} + e\vec{A})^2 \quad (17)$$

where $\vec{A} = \frac{B}{2}(-y, x, 0)$ is the symmetric gauge vector potential.

Landau level wavefunctions

1. Show that the lowest Landau level (LLL) wavefunctions in the symmetric gauge can be written as:

$$\psi_m(z) = \frac{1}{\sqrt{2\pi\ell_B^2 2^m m!}} \left(\frac{z}{\ell_B}\right)^m e^{-|z|^2/4\ell_B^2} \quad (18)$$

where $z = x - iy$, $\ell_B = \sqrt{\hbar c/eB}$ is the magnetic length, and $m = 0, 1, 2, \dots$ is the angular momentum quantum number.

Correction: See class.

2. Verify that these wavefunctions are normalized and orthogonal for different m .

Correction: See class.

3. We define the ‘‘vortex’’ functions

$$\psi(x, y; X, Y) = \frac{1}{\sqrt{2\pi\ell_B^2}} e^{-\frac{|z|^2 + |Z|^2 - 2Z^*z}{4\ell_B^2}}, \quad z = x - iy, \quad Z = X - iY. \quad (19)$$

X and Y are two parameters. Show that they belong to the lowest Landau levels. We will denote the orbital by $|X, Y\rangle$.

Correction: It is trivially an holomorphic function multiplied by the same Gaussian prefactor as our previous basis. Therefore it is in the lowest Landau level.

4. Represent qualitatively $|\psi(x, y; X, Y)|^2$ and $\text{Im} \log \psi(x, y; X, Y)$. Give a physical interpretation of the wavefunctions.

Correction: Let us compute the norm and the argument of the Gaussian factor:

$$\begin{aligned} |z|^2 + |Z|^2 - 2Z^*z &= x^2 + y^2 + X^2 + Y^2 - 2(X + iY)(x - iy) = x^2 + y^2 + X^2 + Y^2 - 2xX - 2yY + 2iYx - 2iXy \\ |z|^2 + |Z|^2 - 2Z^*z &= (x - X)^2 + (y - Y)^2 + 2iYx - 2iXy \end{aligned}$$

We see that the norm of the wavefunction is a Gaussian centred in Z . The phase is simply $R\delta r \sin \delta\theta$, where $\delta\vec{r} = \vec{r} - \vec{R}$ and $\delta\theta$ the angle between \vec{R} and $\delta\vec{r}$. Note that there is no winding/accumulation of the phase, and the function is not a vortex.

5. Show that

$$\langle X', Y' | X, Y \rangle = e^{-\frac{|Z|^2 + |Z'|^2 - 2Z^*Z'}{4\ell_B^2}} \quad (20)$$

Hint: split the integral over x and y . You can use (and/or show using complex analysis) that for $a \in \mathbb{C}$,

$$\int dx e^{-\frac{(x-a)^2}{2}} = \sqrt{2\pi}.$$

Correction: Let us rewrite:

$$\begin{aligned} I &= \frac{1}{2\pi\ell_B^2} e^{-\frac{|Z|^2 + |Z'|^2}{4\ell_B^2}} \iint dx dy e^{-\frac{|z|^2}{2\ell_B^2}} e^{\frac{zZ^* + z^*Z'}{2\ell_B^2}} \\ I &= \frac{1}{2\pi\ell_B^2} e^{-\frac{|Z|^2 + |Z'|^2}{4\ell_B^2}} \iint dx dy e^{-\frac{x^2 + y^2}{2\ell_B^2}} e^{\frac{x(Z^* + Z') + y(iZ^* - iZ')}{2\ell_B^2}} \end{aligned}$$

Now we use

$$\int dx e^{-\frac{x^2}{2}} e^{ax} = e^{\frac{a^2}{2}} \int dx e^{-\frac{(x-a)^2}{2}} = \sqrt{2\pi} e^{\frac{a^2}{2}}$$

even for complex a (proof, introduce $f(z) = e^{-\frac{(z-a)^2}{2}}$, introduce the contour including $z \in \mathbb{R} + a$, and use the absence of pole).

We obtain

$$I = e^{-\frac{|z|^2+|z'|^2}{4l_B^2}} e^{\frac{(z^*+z')^2}{8l_B^2}} e^{-\frac{(z^*-z')^2}{8l_B^2}} = e^{-\frac{|z|^2+|z'|^2}{4l_B^2}} e^{\frac{z^*z'}{2l_B^2}}.$$

6. Bonus (hard): show that, within the lowest Landau level:

$$\int \frac{d\vec{R}}{2\pi l_B^2} |X, Y\rangle \langle X, Y| \quad (21)$$

is the identity operator.

Correction: One way to prove this consists in checking that

$$\int \frac{d\vec{R}}{2\pi l_B^2} \langle m' | X, Y \rangle \langle X, Y | m \rangle = \delta_{m'm}. \quad (22)$$

This is sufficient because the states $|m\rangle$ are known to be a basis for the single-particle states. The expression can be calculated explicitly using the definition of the φ_m and of the vortex functions:

$$I_{mm'} = \frac{1}{(2\pi l_B^2)^2 \sqrt{2^{m+m'} m! m'}} \int \frac{d\vec{R}}{2\pi l_B^2} \int d^2x \int d^2x' \left[e^{-\frac{2|z|^2+|z'|^2-2Zz^*}{4l_B^2}} e^{-\frac{2|z'|^2+|z|^2-2Z^*z'}{4l_B^2}} \right. \\ \left. \times \left(\frac{z'^*}{l_B} \right)^{m'} \left(\frac{z}{l_B} \right)^m \right]. \quad (23)$$

By rotational invariance, only the terms $m = m'$ survive. To see this, we can perform a change of all integration variables \mathbf{x}, \mathbf{x}' , rotating them by the same angle θ . The change of variable cannot modify the result of the integral. However, after the change of variable we have $z \rightarrow ze^{i\theta}$ and $z' \rightarrow z'e^{i\theta}$, $Z \rightarrow Ze^{i\theta}$.

The factors in the exponential are invariant, and only the factors $z^m, z'^{m'}$ transform nontrivially. Thus we have

$$I_{mm'} = e^{i\theta(m-m')} I_{mm'}, \quad (24)$$

for any θ . This is only possible if $I_{mm'} \propto \delta_{mm'}$.

We see therefore that the decomposition of the identity is consistent with rotational invariance (and the conservation of angular momentum). For $m = m'$ we can calculate explicitly:

$$I_{mm} = \frac{1}{(2\pi l_B^2)^2 2^m m!} \int \frac{d\vec{R}}{2\pi l_B^2} \int d^2x \int d^2x' e^{-\frac{|z|^2+|z'|^2+|z|^2-Zz^*-Z^*z'}{2l_B^2}} \left(\frac{zz'^*}{l_B^2} \right)^m \\ = \frac{1}{\pi^3 m!} \int d\vec{R} \int d^2x \int d^2x' e^{-(|z|^2+|z'|^2+|z|^2-Zz^*-Z^*z')} (zz'^*)^m.$$

In the last step, we rescaled variables (in this rescaling the lengths are measured in units of $\sqrt{2}l_B$).

Integrating over \vec{R} gives

$$I_{mm} = \frac{1}{\pi^2 m!} \int d^2x \int d^2x' e^{-(|z|^2+|z'|^2-\frac{1}{4}(z^*+z')^2+\frac{1}{4}(z^*-z')^2)} (zz'^*)^m \\ = \frac{1}{\pi^2 m!} \int d^2x \int d^2x' e^{-(|z|^2+|z'|^2-z^*z')} (zz'^*)^m \\ = \frac{1}{\pi^2 m!} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^2x \int d^2x' e^{-(|z|^2+|z'|^2)} (zz'^*)^m (z^*z')^n.$$

After integrating over polar angles, the only term which is nonzero and which survives is $n = m$. Then we obtain

$$I_{mm} = \frac{1}{\pi^2 (m!)^2} \int d^2x \int d^2x' e^{-(|z|^2 + |z'|^2)} (|zz'^*|)^{2m} = \left(\frac{2}{m!} \int_0^\infty dr r^{2m+1} e^{-r^2} \right)^2 = 1. \quad (27)$$

7. Using the previous answer and your knowledge about bosonic coherent states, what can you say about the vortex states? What volume do they occupy in phase space?

Correction: The vortex basis form an overcomplete basis of the lowest Landau levels. They also occupy a volume $2\pi l_B^2$ in phase space. Following the notations of the class, if we take

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right) + 0 \times \left(b^\dagger b + \frac{1}{2} \right), \quad (28)$$

they correspond to taking a coherent state for the boson b .

Berry connection and curvature

We define the Berry curvature in R -space by

$$A_{X/Y} = i \langle X, Y | \partial_{X/Y} | X, Y \rangle. \quad (29)$$

1. Compute both A_X and A_Y

Correction: By construction,

$$\langle r | \partial_X | X, Y \rangle = \frac{z - X}{2l_B^2} \psi(x, y; X, Y) = \frac{z - Z - iY}{2l_B^2} \psi(x, y; X, Y) \quad (30)$$

$$\langle r | \partial_Y | X, Y \rangle = \frac{iz - Y}{2l_B^2} = \frac{i(z - Z + X)}{2l_B^2} \quad (31)$$

By symmetry, we obtain $A_X = \frac{Y}{2l_B^2}$ and $A_Y = \frac{-X}{2l_B^2}$

2. Deduce the expression of the Berry curvature $\Omega_{XY} = \vec{\nabla}_R \times \vec{A}$.

Correction: Straightforwardly: $\Omega_{XY} = -\frac{1}{l_B^2}$

3. Justify that the Chern number of the lowest Landau level can be written as

$$C = \frac{l_B^2}{S} \int d\vec{R} \Omega_{XY}, \quad (32)$$

where S is the surface of the sample.

4. Deduce the Chern number of the lowest Landau level.

Correction: $C = -1$ trivially.

5. Qualitatively: do you expect the Chern number to change depending the parameters we use to compute it (R-space vs k-space)?

Correction: It should not: it is a global property of the wavefunctions.

6. Qualitatively: do you expect differences in higher Landau levels? Can you explain why?

Correction: Not really. Either we can use the exact mapping between Landau levels, or use the experimental results. The total conductivity increases by 1 for each occupied Landau level, implying $C_{\text{tot}} = -n$, where n is the number of occupied Landau levels. Therefore, each Landau level should contribute -1 .