(A.) The antiferromagnetic ground state

The solution is in the course notes. We copy it here. The vaccuum for the bosons $a^\dagger$, denoted by $|0\rangle$, is the Néel state. This is not the ground state! The ground state obeys $\alpha_{\mathbf{k}} |GS\rangle = 0$. However $\left(u_{\mathbf{k}} a_{\mathbf{k}} + v_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\right)|0\rangle = v_{\mathbf{k}} a^\dagger_{-\mathbf{k}}|0\rangle \neq 0$.

It can be shown that the ground state is:

$$|GS\rangle = \prod_{\mathbf{l}}{}' \frac{1}{u_{\mathbf{l}}} \exp\left(-\frac{v_{\mathbf{l}}}{u_{\mathbf{l}}} a^\dagger_{\mathbf{l}} a^\dagger_{-\mathbf{l}}\right)|0\rangle$$

where $\prod'_{\mathbf{l}}$ means that we do the product on half of the wave vectors, such that only one of each pair $\mathbf{l}$ and $-\mathbf{l}$ is included. In order to ensure this, it must be established that:

$$\alpha_{\mathbf{k}} |GS\rangle = 0$$

with $\alpha_{\mathbf{k}} = u_{\mathbf{k}} a_{\mathbf{k}} + v_{\mathbf{k}} a^\dagger_{-\mathbf{k}}$.

$$\begin{aligned} a_{\mathbf{k}}|GS\rangle &= a_{\mathbf{k}} \prod_{\mathbf{l}}{}' \frac{1}{u_{\mathbf{l}}} \exp\left(-\frac{v_{\mathbf{l}}}{u_{\mathbf{l}}} a^\dagger_{\mathbf{l}} a^\dagger_{-\mathbf{l}}\right)|0\rangle \\ &= \prod_{\mathbf{l}\neq \mathbf{k},-\mathbf{k}}{}' \frac{1}{u_{\mathbf{l}}} \exp\left(-\frac{v_{\mathbf{l}}}{u_{\mathbf{l}}} a^\dagger_{\mathbf{l}} a^\dagger_{-\mathbf{l}}\right) \frac{1}{u_{\mathbf{k}}} a_{\mathbf{k}} \sum_{n=1}^{+\infty} \frac{\left[-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\right]^n}{n!}|0\rangle \end{aligned}$$

where the sum starts at $n=1$ because the term $n=0$ gives 0 as $a_{\mathbf{k}}|0\rangle = 0$. In addition, we have

$$[a_{\mathbf{k}}, (a^\dagger_{\mathbf{k}})^n] = n(a^\dagger_{\mathbf{k}})^{n-1}$$

Let us show it by induction. This is true for $n=1$ since $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}] = 1$. Suppose this is true at the order $n-1$. It follows:

$$\begin{aligned} [a_{\mathbf{k}}, (a^\dagger_{\mathbf{k}})^n] &= [a_{\mathbf{k}}, (a^\dagger_{\mathbf{k}})^{n-1}] a^\dagger_{\mathbf{k}} + (a^\dagger_{\mathbf{k}})^{n-1}[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}] \\ &= (n-1)(a^\dagger_{\mathbf{k}})^{n-2} a^\dagger_{\mathbf{k}} + (a^\dagger_{\mathbf{k}})^{n-1} \\ &= n(a^\dagger_{\mathbf{k}})^{n-1} \end{aligned}$$

Therefore,

$$a_{\mathbf{k}}(a^\dagger_{\mathbf{k}})^n = n(a^\dagger_{\mathbf{k}})^{n-1} + \underbrace{(a^\dagger_{\mathbf{k}})^n a_{\mathbf{k}}}_{\text{gives 0 on the vaccuum}}$$

and we get:

$$\begin{aligned} a_{\mathbf{k}}|GS\rangle &= \prod_{\mathbf{l}\neq \mathbf{k},-\mathbf{k}}{}' \frac{1}{u_{\mathbf{l}}} \exp\left(-\frac{v_{\mathbf{l}}}{u_{\mathbf{l}}} a^\dagger_{\mathbf{l}} a^\dagger_{-\mathbf{l}}\right) \frac{1}{u_{\mathbf{k}}} \sum_{n=1}^{+\infty} \frac{(-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}})^n (a^\dagger_{\mathbf{k}})^{n-1} (a^\dagger_{-\mathbf{k}})^n}{(n-1)!}|0\rangle \\ &= \prod_{\mathbf{l}\neq \mathbf{k},-\mathbf{k}}{}' \frac{1}{u_{\mathbf{l}}} \exp\left(-\frac{v_{\mathbf{l}}}{u_{\mathbf{l}}} a^\dagger_{\mathbf{l}} a^\dagger_{-\mathbf{l}}\right) \frac{1}{u_{\mathbf{k}}} \frac{-v_{\mathbf{k}}}{u_{\mathbf{k}}} a^\dagger_{-\mathbf{k}} \exp\left(-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\right)|0\rangle \end{aligned}$$

Finally:

$$u_{\mathbf{k}} a_{\mathbf{k}} |GS\rangle = -v_{\mathbf{k}} a^\dagger_{-\mathbf{k}} |GS\rangle$$

and thus,

$$\alpha_{\mathbf{k}} |GS\rangle = 0.$$

Lastly, let us check that $|F\rangle$ is normalized.

$$\begin{aligned}
\langle GS|GS\rangle &= \langle 0| \prod_{\mathbf{k}}{}' \frac{1}{u_{\mathbf{k}}^2} \sum_n \left(-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}\right)^{2n} \frac{(a_{-\mathbf{k}} a_{\mathbf{k}})^n (a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}})^n}{n!^2} |0\rangle \\
&= \prod_{\mathbf{k}}{}' \frac{1}{u_{\mathbf{k}}^2} \sum_n \left(-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}\right)^{2n} \\
&= \prod_{\mathbf{k}}{}' \frac{1}{u_{\mathbf{k}}^2} \frac{1}{1-(\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}})^2} \\
&= 1.
\end{aligned}$$

(B.) Correction to the magnetization in the antiferromagnetic case

Consider the dispersion relation

$$\omega_k = 6JS\sqrt{1-\gamma_{\mathbf{k}}^2}, \quad \gamma_{\mathbf{k}} = \frac{1}{3}(\cos k_x + \cos k_y + \cos k_z)$$

1. $\omega_{\mathbf{k}} = 0 \Leftrightarrow |\gamma_{\mathbf{k}}| = 1 \Leftrightarrow k_x = k_y = k_z = 0, \pi$. For $\mathbf{k} \to 0$

$$\begin{aligned}
\cos k_x &\simeq 1 - \frac{1}{2}k_x^2 \\
\Rightarrow \omega_{\mathbf{k}} &\simeq 6JS\sqrt{1-\frac{1}{9}(3-\frac{1}{2}\mathbf{k}^2)^2} \\
&\simeq 6JS\sqrt{1-\frac{1}{9}(9-3\mathbf{k}^2)} \\
&\simeq 6JS\sqrt{\frac{1}{3}\mathbf{k}^2} \\
&= 2\sqrt{3}JS|\mathbf{k}|
\end{aligned}$$

For $\boldsymbol{q} = (\pi,\pi,\pi)$ et $\mathbf{k} \to 0$

$$\begin{aligned}
\cos(q_x + k_x) &= -\cos(k_x) \\
\Rightarrow \gamma_{\boldsymbol{q}+\mathbf{k}} &= -\gamma_{\mathbf{k}} \\
\Rightarrow \omega_{\boldsymbol{q}+\mathbf{k}} &= \omega_{\mathbf{k}} \\
&\simeq 2\sqrt{3}JS|\mathbf{k}|
\end{aligned}$$

2. We must evaluate

$$\delta M^{(2)} = -\frac{1}{N}\sum_{\mathbf{k}} \frac{u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2}{e^{\frac{\omega_{\mathbf{k}}}{T}} - 1}$$

In the thermodynamic limit, k becomes a continuous variable. We can therefore replace the sum by an integral on the first Brillouin zone ($-\pi < k_x, k_y, k_z \leq \pi$):

$$\delta M^{(2)} \simeq -\frac{1}{(2\pi)^3} \int \frac{u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2}{e^{\frac{\omega_{\mathbf{k}}}{T}} - 1} d^3k$$

But $\omega_{\boldsymbol{q}+\mathbf{k}} = \omega_{\mathbf{k}}$ for $\boldsymbol{q} = (\pm\pi, \pm\pi, \pm\pi)$

$$\Rightarrow u_{\boldsymbol{q}+\mathbf{k}}^2 = u_{\mathbf{k}}^2 \text{ and } v_{\boldsymbol{q}+\mathbf{k}}^2 = v_{\mathbf{k}}^2$$

We can therefore restrict the integration domain to $\mathcal{D} = [-\pi, \pi] \times [-\pi, \pi] \times [-\pi/2, \pi/2]$ (one half of the cube of side π):

$$\delta M^{(2)} \simeq -\frac{2}{(2\pi)^3} \int_{\mathcal{D}} \frac{u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2}{e^{\frac{\omega_{\mathbf{k}}}{T}} - 1} d^3k$$

For low temperatures ($T \ll JS$), the Bose-Einstein distribution diverges when $\omega_{\mathbf{k}} = 0$. For the considered integration domain $\omega_{\boldsymbol{k}} = 0$ only at $\mathbf{k} = 0$. We thus infer that the terms that dominate in the integrals are those with small wave vectors. To capture the dependency in T we can then use spherical coordinates and integrate on a sphere centered in $\mathbf{k} = 0$, with radius δ.

$$\delta M^{(2)} \simeq -\frac{1}{\pi^2} \int_0^{\delta} k^2 \frac{u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2}{e^{\frac{\omega_{\mathbf{k}}}{T}} - 1} dk$$

Using the expansion of $\omega_{\mathbf{k}}$ for $\mathbf{k} \to 0$ we have

$$\begin{aligned} u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 &= \frac{6JS}{\omega_{\mathbf{k}}} \\ &\simeq \frac{\sqrt{3}}{k} \end{aligned}$$

and

$$\delta M^{(2)} \simeq -\frac{\sqrt{3}}{\pi^2} \int_0^{\delta} \frac{k}{e^{\frac{2\sqrt{3}JSk}{T}} - 1} dk$$

We use the change of variable $p = \frac{2\sqrt{3}JS}{T} k$

$$\begin{aligned} \delta M^{(2)} &\simeq -\frac{1}{4\sqrt{3}\pi^2} \left(\frac{T}{JS}\right)^2 \underbrace{\int_0^{\infty} \frac{p}{e^p - 1} dp}_{<\infty} \\ &\sim -\left(\frac{T}{JS}\right)^2 \end{aligned}$$

3. Likewise,

$$\begin{aligned} E &\simeq \int \omega_{\mathbf{k}} n_{\mathbf{k}} d^3k \\ &= 2 \int_{\mathcal{D}} \omega_{\mathbf{k}} n_{\mathbf{k}} d^3k \\ &\simeq 8\pi \int_0^{\delta} \frac{k^2 \omega_{\mathbf{k}}}{e^{\frac{\omega_{\mathbf{k}}}{T}} - 1} dk \\ &\sim JS \left(\frac{T}{JS}\right)^4 \underbrace{\int_0^{\infty} \frac{p^3}{e^p - 1} dp}_{<\infty} \end{aligned}$$

and

$$
\begin{aligned}
c^{\mathrm{AF}}_{\mathrm{magnetic}} &= \frac{\partial E}{\partial T} \\
&\sim \left(\frac{T}{JS}\right)^3
\end{aligned}
$$