Solution session 12

1. We have

$$\begin{aligned} |\Phi_N\rangle &= C\Big(\sum_{\boldsymbol{k}} g_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}\Big)^{N/2}|0\rangle \\ &= C \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_{N/2}} \big(g_{\boldsymbol{k}_1}\ldots g_{\boldsymbol{k}_{N/2}}\big)\, c^\dagger_{\boldsymbol{k}_1\uparrow} c^\dagger_{-\boldsymbol{k}_1\downarrow}\cdots c^\dagger_{\boldsymbol{k}_{N/2}\uparrow} c^\dagger_{-\boldsymbol{k}_{N/2}\downarrow}|0\rangle \\ &= C \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_{N/2}} \big(g_{\boldsymbol{k}_1}\ldots g_{\boldsymbol{k}_{N/2}}\big)\, |\boldsymbol{k}_1\uparrow,-\boldsymbol{k}_1\downarrow,\ldots,\boldsymbol{k}_{N/2}\uparrow,-\boldsymbol{k}_{N/2}\downarrow\rangle \end{aligned}$$

and therefore:

$$\langle \boldsymbol{r}_1\sigma_1,\boldsymbol{r}_2\sigma_2,\ldots \boldsymbol{r}_N\sigma_N|\Phi_N\rangle = C \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_{N/2}} \big(g_{\boldsymbol{k}_1}\ldots g_{\boldsymbol{k}_{N/2}}\big) \sum_p s_p$$
$$\times \langle \boldsymbol{r}_{p_1}\sigma_{p_1}|\boldsymbol{k}_1\uparrow\rangle\,\langle \boldsymbol{r}_{p_2}\sigma_{p_2}|-\boldsymbol{k}_1\downarrow\rangle\ldots\langle \boldsymbol{r}_{p_{N-1}}\sigma_{p_{N-1}}|\boldsymbol{k}_{N/2}\uparrow\rangle\,\langle \boldsymbol{r}_{p_N}\sigma_{p_N}|-\boldsymbol{k}_{N/2}\downarrow\rangle$$

where $p_i \equiv p(i)$. Here we used the fact that

$$\begin{aligned} \langle \phi_1,\ldots,\phi_N|\varphi_1,\ldots,\varphi_N\rangle &= \frac{1}{N!}\sum_{p,q\in S_N} \zeta^{p+q}\langle\phi_{p(1)}|\varphi_{q(1)}\rangle\cdot\cdots\cdot\langle\phi_{p(N)}|\varphi_{q(N)}\rangle \\ &= \frac{1}{N!}\sum_{p,q\in S_N} \zeta^{p+q}\langle\phi_{p\cdot q^{-1}(1)}|\varphi_1\rangle\cdot\cdots\cdot\langle\phi_{p\cdot q^{-1}(N)}|\varphi_N\rangle \\ &= \sum_{r\in S_N}\zeta^r\langle\phi_{r(1)}|\varphi_1\rangle\cdot\cdots\cdot\langle\phi_{r(N)}|\varphi_N\rangle. \end{aligned}$$

Using the formula $\langle \boldsymbol{r}\sigma|\boldsymbol{k}\sigma'\rangle = \delta_{\sigma\sigma'}\frac{1}{\sqrt{\Omega}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \eta_{\sigma'}(\sigma)\frac{1}{\sqrt{\Omega}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, we obtain

$$\begin{aligned} \langle \boldsymbol{r}_1\sigma_1,\boldsymbol{r}_2\sigma_2,\ldots \boldsymbol{r}_N\sigma_N|\Phi_N\rangle &= C \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_{N/2}} \big(g_{\boldsymbol{k}_1}\ldots g_{\boldsymbol{k}_{N/2}}\big) \sum_p s_p \\ &\quad\times \frac{1}{\sqrt{\Omega}}e^{i\boldsymbol{k}_1\cdot\boldsymbol{r}_{p_1}}\eta_\uparrow(\sigma_{p_1})\frac{1}{\sqrt{\Omega}}e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_{p_2}}\eta_\downarrow(\sigma_{p_2}) \\ &\quad\vdots \\ &\quad\times \frac{1}{\sqrt{\Omega}}e^{i\boldsymbol{k}_{N/2}\cdot\boldsymbol{r}_{p_{N-1}}}\eta_\uparrow(\sigma_{p_{N-1}})\frac{1}{\sqrt{\Omega}}e^{-i\boldsymbol{k}_{N/2}\cdot\boldsymbol{r}_{p_{N-1}}}\eta_\downarrow(\sigma_{p_N}) \end{aligned}$$

so that

$$\begin{aligned}
\langle \boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2, \dots \boldsymbol{r}_N\sigma_N|\Phi_N\rangle &= C\sum_p s_p \\
&\times \frac{1}{\Omega}\sum_{\boldsymbol{k}_1} g_{\boldsymbol{k}_1} e^{i\boldsymbol{k}_1\cdot(\boldsymbol{r}_{p_1}-\boldsymbol{r}_{p_2})}\eta_\uparrow(\sigma_{p_1})\eta_\downarrow(\sigma_{p_2}) \\
&\times \frac{1}{\Omega}\sum_{\boldsymbol{k}_2} g_{\boldsymbol{k}_2} e^{i\boldsymbol{k}_2\cdot(\boldsymbol{r}_{p_3}-\boldsymbol{r}_{p_4})}\eta_\uparrow(\sigma_{p_3})\eta_\downarrow(\sigma_{p_4}) \\
&\vdots \\
&\times \frac{1}{\Omega}\sum_{\boldsymbol{k}_{N/2}} g_{\boldsymbol{k}_{N/2}} e^{i\boldsymbol{k}_{N/2}\cdot(\boldsymbol{r}_{p_{N-1}}-\boldsymbol{r}_{p_N})}\eta_\uparrow(\sigma_{p_{N-1}})\eta_\downarrow(\sigma_{p_N}) \\
&= \frac{C}{\Omega^{N/4}}\sum_p s_p \psi(\boldsymbol{r}_{p_1}\sigma_{p_1}, \boldsymbol{r}_{p_2}\sigma_{p_2})\psi(\boldsymbol{r}_{p_3}\sigma_{p_3}, \boldsymbol{r}_{p_4}\sigma_{p_4})\dots\psi(\boldsymbol{r}_{p_{N-1}}\sigma_{p_{N-1}}, \boldsymbol{r}_{p_N}\sigma_{p_N}) \\
\Rightarrow \langle \boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2, \dots \boldsymbol{r}_N\sigma_N|\Phi_N\rangle &\propto \mathcal{A}\cdot\Big(\psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)\psi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\dots\psi(\boldsymbol{r}_{N-1}\sigma_{N-1}, \boldsymbol{r}_N\sigma_N)\Big)
\end{aligned}$$

2. We can antisymmetrise the two-particle wave function. Indeed, using

$$\mathcal{A}\cdot f(\cdots, \boldsymbol{r}_i\sigma_i, \boldsymbol{r}_j\sigma_j, \cdots) = -\mathcal{A}\cdot f(\cdots, \boldsymbol{r}_j\sigma_j, \boldsymbol{r}_i\sigma_i, \cdots)$$

we have

$$\begin{aligned}
\mathcal{A}\cdot f(\cdots, \boldsymbol{r}_i\sigma_i, \boldsymbol{r}_j\sigma_j, \cdots) &= \frac{1}{2}\Big(\mathcal{A}\cdot f(\cdots, \boldsymbol{r}_i\sigma_i, \boldsymbol{r}_j\sigma_j, \cdots) - \mathcal{A}\cdot f(\cdots, \boldsymbol{r}_j\sigma_j, \boldsymbol{r}_i\sigma_i, \cdots)\Big) \\
&= \mathcal{A}\cdot\frac{1}{2}\Big(f(\cdots, \boldsymbol{r}_i\sigma_i, \boldsymbol{r}_j\sigma_j, \cdots) - f(\cdots, \boldsymbol{r}_j\sigma_j, \boldsymbol{r}_i\sigma_i, \cdots)\Big)
\end{aligned}$$

Hence, for particles 1 and 2,

$$\begin{aligned}
\langle \boldsymbol{r}_1\sigma_1, \cdots, \boldsymbol{r}_N\sigma_N|\Phi_N\rangle &\propto \mathcal{A}\cdot\Big(\psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)\psi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\cdots\Big) \\
&= \frac{1}{2}\mathcal{A}\cdot\Big(\psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)\psi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\cdots\psi(\boldsymbol{r}_{N-1}\sigma_{N-1}, \boldsymbol{r}_N\sigma_N) \\
&\quad -\psi(\boldsymbol{r}_2\sigma_2, \boldsymbol{r}_1\sigma_1)\psi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\cdots\psi(\boldsymbol{r}_{N-1}\sigma_{N-1}, \boldsymbol{r}_N\sigma_N)\Big) \\
&= \frac{1}{\sqrt{2}}\mathcal{A}\cdot\Big(\frac{\psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)-\psi(\boldsymbol{r}_2\sigma_2, \boldsymbol{r}_1\sigma_1)}{\sqrt{2}} \\
&\quad \times\psi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\cdots\psi(\boldsymbol{r}_{N-1}\sigma_{N-1}, \boldsymbol{r}_N\sigma_N)\Big)
\end{aligned}$$

By performing the same operation for the pairs of particles $(3,4), (5,6), \cdots, (N-1, N)$, we obtain

$$\langle \boldsymbol{r}_1\sigma_1, \cdots, \boldsymbol{r}_N\sigma_N|\Phi_N\rangle \propto \mathcal{A}\cdot\Big(\psi_a(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)\psi_a(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\cdots\psi_a(\boldsymbol{r}_{N-1}\sigma_{N-1}, \boldsymbol{r}_N\sigma_N)\Big)$$

with the antisymmetrised two-particle wave function

$$\begin{aligned}
\psi_a(\boldsymbol{r}\sigma, \boldsymbol{r}'\sigma') &= \frac{\psi(\boldsymbol{r}\sigma, \boldsymbol{r}'\sigma') - \psi(\boldsymbol{r}'\sigma', \boldsymbol{r}\sigma)}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\left(\varphi(\boldsymbol{r}-\boldsymbol{r}')\eta_\uparrow(\sigma)\eta_\downarrow(\sigma') - \underbrace{\varphi(\boldsymbol{r}'-\boldsymbol{r})}_{=\varphi(\boldsymbol{r}-\boldsymbol{r}')}\eta_\uparrow(\sigma')\eta_\downarrow(\sigma)\right) \\
&= \frac{1}{\sqrt{2}}\varphi(\boldsymbol{r}-\boldsymbol{r}')\Big(\eta_\uparrow(\sigma)\eta_\downarrow(\sigma') - \eta_\downarrow(\sigma)\eta_\uparrow(\sigma')\Big)
\end{aligned}$$

3. We must evaluate

$$\xi^2 \simeq \frac{\langle\psi|\hat{R}^2|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int \varphi^*(\boldsymbol{R})R^2\varphi(\boldsymbol{R})d^3R}{\int \varphi^*(\boldsymbol{R})\varphi(\boldsymbol{R})d^3R}$$

with

$$\varphi(\boldsymbol{R}) = \frac{1}{\sqrt{\Omega}}\sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{R}}$$

and $g_{\boldsymbol{k}} = \frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}}$, $u_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1+\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$, $v_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1-\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$, a and $E_{\boldsymbol{k}} = \sqrt{\xi_{\boldsymbol{k}}^2+\Delta^2}$.
Let us start with the numerator. Switching to the continuous limit for $\boldsymbol{k}$

$$\begin{aligned}\int \varphi^*(\boldsymbol{R})R^2\varphi(\boldsymbol{R})d^3R &= \frac{\Omega}{(2\pi)^6}\int d^3R \int d^3k g_{\boldsymbol{k}}\boldsymbol{R}e^{i\boldsymbol{k}\cdot\boldsymbol{R}} \cdot \int d^3p g_{\boldsymbol{p}}\boldsymbol{R}e^{-i\boldsymbol{p}\cdot\boldsymbol{R}} \\ &= \frac{\Omega}{(2\pi)^6}\int d^3R \int d^3k g_{\boldsymbol{k}}\boldsymbol{\nabla}_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{R}} \cdot \int d^3p g_{\boldsymbol{p}}\boldsymbol{\nabla}_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{R}}\end{aligned}$$

However, $\nabla_{\boldsymbol{k}}(f(\boldsymbol{k})g(\boldsymbol{k})) = f(\boldsymbol{k})\nabla_{\boldsymbol{k}}g(\boldsymbol{k}) + g(\boldsymbol{k})\nabla_{\boldsymbol{k}}f(\boldsymbol{k})$. Therefore we have:

$$\int d^3k g_{\boldsymbol{k}}\boldsymbol{\nabla}_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{R}} = \int d^3k \boldsymbol{\nabla}_{\boldsymbol{k}}\left(g_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\right) - \int d^3k(\boldsymbol{\nabla}_{\boldsymbol{k}}g_{\boldsymbol{k}})e^{i\boldsymbol{k}\cdot\boldsymbol{R}}$$

We then use Gauss's theorem (the divergence theorem) on the first term

$$\int d^3k \boldsymbol{\nabla}_{\boldsymbol{k}}\left(g_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\right) = \int dS g_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\boldsymbol{e_r}$$

We need to integrate over a sphere of radius $|\boldsymbol{k}| \to \infty$. Yet, $g_{\boldsymbol{k}} \to 0$ when $\xi_k > \xi_{k_F} + \omega_D$. More precisely, recall the expression of g_k, u_k and v_k (recall that $\xi_k = \frac{k^2}{2m} - \epsilon_F$). You see that for very large k, $E_k \simeq \xi_k + \frac{1}{2}\frac{\Delta^2}{\xi_k}$. By expanding, one gets that

$$g_{\boldsymbol{k}} = \frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}} \simeq \frac{m\Delta}{\boldsymbol{k}^2} \tag{1}$$

Since the surface $dS \simeq 4\pi k^2$, we are essentially integrating $e^{i\boldsymbol{k}\cdot\boldsymbol{R}}$ on the sphere, which gives zero. We thus have

$$\int d^3k \boldsymbol{\nabla}_{\boldsymbol{k}}\left(g_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\right) = 0$$

and

$$\begin{aligned}\int \varphi^*(\boldsymbol{R})R^2\varphi(\boldsymbol{R})d^3R &= \frac{\Omega}{(2\pi)^6}\int d^3R \int d^3k g_{\boldsymbol{k}}\boldsymbol{R}e^{i\boldsymbol{k}\cdot\boldsymbol{R}} \cdot \int d^3p g_{\boldsymbol{p}}\boldsymbol{R}e^{-i\boldsymbol{p}\cdot\boldsymbol{R}} \\ &= \frac{\Omega}{(2\pi)^6}\int d^3R \int d^3k (\boldsymbol{\nabla}_{\boldsymbol{k}}g_{\boldsymbol{k}})e^{i\boldsymbol{k}\cdot\boldsymbol{R}} \cdot \int d^3p (\boldsymbol{\nabla}_{\boldsymbol{p}}g_{\boldsymbol{p}})e^{-i\boldsymbol{p}\cdot\boldsymbol{R}} \\ &= \frac{\Omega}{(2\pi)^6}\int d^3k \int d^3p (\boldsymbol{\nabla}_{\boldsymbol{k}}g_{\boldsymbol{k}}) \cdot (\boldsymbol{\nabla}_{\boldsymbol{p}}g_{\boldsymbol{p}}) \underbrace{\int e^{i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{R}}d^3R}_{=(2\pi)^3\delta_{\boldsymbol{k},\boldsymbol{p}}} \\ &= \frac{\Omega}{(2\pi)^3}\int d^3k (\boldsymbol{\nabla}_{\boldsymbol{k}}g_{\boldsymbol{k}})^2\end{aligned}$$

At this point, two remarks are needed:
Remember that g_k behaves as

- We consider a set of Cooper pairs in the presence of a a Fermi sea. Therefore, for $\epsilon_{\boldsymbol{k}} < \epsilon_F - \omega_D$, $g_{\boldsymbol{k}}$ is constant and its gradient it zero.
- $\boldsymbol{\nabla}_{\boldsymbol{k}} g_{\boldsymbol{k}}$ only depends on $|\boldsymbol{k}|$, so that we can switch to spherical coordinates and integrate over the angular variables.

$$\begin{aligned}
\int \varphi^*(\boldsymbol{R}) R^2 \varphi(\boldsymbol{R}) d^3R &= \frac{\Omega}{2\pi^2} \int_{k_F}^{k_0} dk k^2 \left(\frac{dg_k}{dk}\right)^2 \\
&\simeq \frac{\Omega}{2\pi^2} k_F^2 \int_{k_F}^{k_0} dk \left(\frac{dg_k}{dk}\right)^2 \\
&= \frac{\Omega}{2\pi^2} k_F^2 \int_0^{\omega_D} d\xi_k \left(\frac{dk}{d\xi_k}\right) \left(\frac{d\xi_k}{dk}\frac{dg_k}{d\xi_k}\right)^2 \\
&\simeq \frac{\Omega}{2\pi^2} k_F^2 \left(\frac{d\xi_k}{dk}\right)_{\xi_k=0} \int_0^{\omega_D} d\xi_k \left(\frac{dg_k}{d\xi_k}\right)^2
\end{aligned}$$

With

$$g_{\boldsymbol{k}} = \sqrt{\frac{E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}} + \xi_{\boldsymbol{k}}}} = \sqrt{\frac{E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}} + \xi_{\boldsymbol{k}}}}\sqrt{\frac{E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}}} = \frac{E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}}{\sqrt{E_{\boldsymbol{k}}^2 - \xi_{\boldsymbol{k}}^2}} = \frac{1}{\Delta}\left(E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}\right)$$

we have

$$\begin{aligned}
\left(\frac{dg_k}{d\xi_k}\right)^2 &= \frac{1}{\Delta^2}\left(\frac{d}{d\xi_k}\left(E - \xi_k\right)\right)^2 \\
&= \frac{1}{\Delta^2}\left(\frac{d}{d\xi_k}\left(\sqrt{\xi_k^2 + \Delta^2} - \xi_k\right)\right)^2 \\
&= \frac{1}{\Delta^2}\left(1 + \frac{\xi_k^2}{\xi_k^2 + \Delta^2} - 2\frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\omega_D} d\xi_k \left(\frac{dg_k}{d\xi_k}\right)^2 &= \frac{1}{\Delta^2}\left[\xi_k + (\xi_k - \Delta \arctan\frac{\xi_k}{\Delta}) - 2\sqrt{\xi_k^2 + \Delta^2}\right]_0^{\omega_D} \\
&= \frac{1}{\Delta^2}\left(2\omega_D - \Delta \arctan\frac{\omega_D}{\Delta} + 2\sqrt{\omega_D^2 + \Delta^2} + 2\Delta\right)
\end{aligned}$$

But $\Delta = 2\omega_D e^{\frac{-1}{V g_F}}$ (see course), so that in the weak coupling limit $\Delta \ll \omega_D \Rightarrow \sqrt{\omega_D^2 + \Delta^2} \simeq \omega_D$ and

$$\int_0^{\omega_D} d\xi_k \left(\frac{dg_k}{d\xi_k}\right)^2 \simeq \frac{2 - \frac{\pi}{2}}{\Delta}$$

Thus we have, with $\left(\frac{d\xi_k}{dk}\right)_{\xi_k=0} = \hbar v_F$,

$$\int \varphi^*(\boldsymbol{R}) R^2 \varphi(\boldsymbol{R}) d^3R = \frac{\Omega k_F^2 \hbar v_F}{\pi^2 \Delta}\left(1 - \frac{\pi}{4}\right)$$

For the denominator

$$\begin{aligned}\int \varphi^*(\boldsymbol{R})\varphi(\boldsymbol{R})d^3R &\simeq \frac{\Omega}{(2\pi)^6}\int d^3k \int d^3p g_k g_p \int d^3R e^{i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{R}} \\ &= \frac{\Omega}{(2\pi)^3}\int d^3k \int d^3p g_{\boldsymbol{k}} g_{\boldsymbol{p}} \delta(\boldsymbol{k}-\boldsymbol{p}) \\ &= \frac{\Omega}{(2\pi)^3}\int d^3k g_{\boldsymbol{k}}^2\end{aligned}$$

We use spherical coordinates and integrate over the angular variables

$$\begin{aligned}\int \varphi^*(\boldsymbol{R})\varphi(\boldsymbol{R})d^3R &\simeq \frac{\Omega}{2\pi^2}k_F^2\int_{k_F}^{k_0} dk g_k^2 \\ &\simeq \frac{\Omega}{2\pi^2}k_F^2\left(\frac{dk}{d\xi_k}\right)_{\xi_k=0}\int_0^{\omega_D} g_k^2 d\xi_k \\ &= \frac{\Omega}{2\pi^2}\frac{k_F^2}{\hbar v_F}\int_0^{\omega_D}\left(\frac{\sqrt{\Delta^2+\xi_k^2}-\xi_k}{\Delta}\right)^2 d\xi_k \\ &= \frac{\Omega}{2\pi^2}\frac{k_F^2}{\hbar v_F\Delta^2}\int_0^{\omega_D}\left(\Delta^2+2\xi_k^2-2\xi_k\sqrt{\Delta^2+\xi_k^2}\right)d\xi_k \\ &= \frac{\Omega}{2\pi^2}\frac{k_F^2}{\hbar v_F\Delta^2}\left[\xi_k\Delta^2+\frac{2}{3}\xi_k^3-\frac{2}{3}(\Delta^2+\xi_k^2)\sqrt{\Delta^2+\xi_k^2}\right]_0^{\omega_D} \\ &= \frac{\Omega}{2\pi^2}\frac{k_F^2}{\hbar v_F\Delta^2}\left(\omega_D\Delta^2+\frac{2}{3}\omega_D^3-\frac{2}{3}(\Delta^2+\omega_D^2)\sqrt{\Delta^2+\omega_D^2}+\frac{2}{3}\Delta^3\right)\end{aligned}$$

But $\Delta \ll \omega_D \Rightarrow \sqrt{1+\left(\frac{\Delta}{\omega_D}\right)^2} \simeq 1+\Delta^2/(2\omega_D^2)$ and

$$\int \varphi^*(\boldsymbol{R})\varphi(\boldsymbol{R})d^3R \simeq \frac{\Omega k_F^2\Delta}{3\pi^2\hbar v_F}$$

Finally,

$$\ell_c^2 \simeq \frac{\int \varphi^*(\boldsymbol{R})R^2\varphi(\boldsymbol{R})d^3R}{\int \varphi^*(\boldsymbol{R})\varphi(\boldsymbol{R})d^3R} \sim \frac{\hbar^2 v_F^2}{\Delta^2}$$