BCS Hamiltonian:

$$H_{BCS} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \frac{|V|}{\Omega} \sum_{\mathbf{k}\neq\mathbf{p}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \tag{1}$$

BCS variational function:

$$|\Psi_{BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle \tag{2}$$

(A.) How to determine $\langle \Psi_{BCS} | \Psi_{BCS} \rangle$. In noting that $[c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow}] = 0$ if $\mathbf{k} \neq \mathbf{p}$, we have

$$\begin{aligned}
\langle \Psi_{BCS} | \Psi_{BCS} \rangle &= \langle 0 | \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) \prod_{\mathbf{p}} (u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow}) |0\rangle \\
&= \langle 0 | \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow})(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) |0\rangle \\
&= \langle 0 | \prod_{\mathbf{k}} \left(u_{\mathbf{k}}^2 + u_{\mathbf{k}} v_{\mathbf{k}} \underbrace{c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}}_{\to 0} + u_{\mathbf{k}} v_{\mathbf{k}} \underbrace{c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}}_{\to 0} + v_{\mathbf{k}}^2 c_{-\mathbf{k}\downarrow} \underbrace{c_{\mathbf{k}\uparrow} c^\dagger_{\mathbf{k}\uparrow}}_{1 - c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow}} c^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle \\
&= \langle 0 | \prod_{\mathbf{k}} \left(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 \underbrace{c_{-\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow}}_{1 - c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow}} + v_{\mathbf{k}}^2 \underbrace{c^\dagger_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}}_{\to 0} \right) |0\rangle \\
&= \prod_{\mathbf{k}} \left(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 \right)
\end{aligned}$$

So by choosing $u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$, we have

$$\langle \Psi_{BCS} | \Psi_{BCS} \rangle = 1 \tag{3}$$

(B.) To determine $\langle H_{BCS} \rangle \equiv \langle \Psi_{BCS} | H_{BCS} | \Psi_{BCS} \rangle$, begin with the kinetic part:

$$\sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} \langle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \rangle \tag{4}$$

For $\sigma =\uparrow$

$$\begin{aligned}
\langle c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow}\rangle &= \langle 0|\prod_{\boldsymbol{q}}(u_{\boldsymbol{q}} + v_{\boldsymbol{q}} c_{-\boldsymbol{q}\downarrow} c_{\boldsymbol{q}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} \prod_{\mathbf{p}}(u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow})|0\rangle \\
&= \langle 0| \underbrace{\left(\prod_{\boldsymbol{p}\neq\boldsymbol{k}}(u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow})(u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow})\right)}_{\prod_{\boldsymbol{p}\neq\boldsymbol{k}}(u^2_{\mathbf{p}}+v^2_{\mathbf{p}})=1} \\
&\quad \times (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \\
&= \langle 0|(u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle
\end{aligned}$$

more

$$\begin{aligned}
c_{\mathbf{k}\uparrow}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle &= v_{\mathbf{k}} c_{\mathbf{k}\uparrow} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}|0\rangle \\
&= v_{\mathbf{k}}(1 - c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow}) c^\dagger_{-\mathbf{k}\downarrow}|0\rangle \\
&= v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}|0\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle 0|(u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} &= \left(c_{\mathbf{k}\uparrow}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle\right)^\dagger \\
&= v_{\mathbf{k}}\langle 0|c_{-\boldsymbol{k}\downarrow}
\end{aligned}$$

from where

$$\begin{aligned}
\langle 0|(u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle &= v^2_{\mathbf{k}}\langle 0|c_{-\boldsymbol{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow}|0\rangle \\
&= v^2_{\mathbf{k}}
\end{aligned}$$

Likewise for $\sigma =\downarrow$

$$\begin{aligned}
\langle c^\dagger_{\mathbf{k}\downarrow} c_{\mathbf{k}\downarrow}\rangle &= \langle 0|\prod_{\boldsymbol{q}}(u_{\boldsymbol{q}} + v_{\boldsymbol{q}} c_{-\boldsymbol{q}\downarrow} c_{\boldsymbol{q}\uparrow}) c^\dagger_{\mathbf{k}\downarrow} c_{\mathbf{k}\downarrow} \prod_{\mathbf{p}}(u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow})|0\rangle \\
&= \langle 0| \underbrace{\left(\prod_{\boldsymbol{p}\neq-\boldsymbol{k}}(u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow})(u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow})\right)}_{\prod_{\boldsymbol{p}\neq-\boldsymbol{k}}(u^2_{\mathbf{p}}+v^2_{\mathbf{p}})=1} \\
&\quad \times (u_{-\boldsymbol{k}} + v_{-\mathbf{k}} c_{\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow}) c^\dagger_{\mathbf{k}\downarrow} c_{\mathbf{k}\downarrow} (u_{-\mathbf{k}} + v_{-\mathbf{k}} c^\dagger_{-\mathbf{k}\uparrow} c^\dagger_{\mathbf{k}\downarrow})|0\rangle \\
&\ \ \vdots \\
&= v^2_{-\mathbf{k}}
\end{aligned}$$

Finally, noticing that $\xi_{-\boldsymbol{k}} = \xi_{\mathbf{k}}$,

$$\sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} \langle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}\rangle = 2\sum_{\mathbf{k}} \xi_{\mathbf{k}} v^2_{\mathbf{k}} \tag{5}$$

Let's now consider the interaction term:

$$-\frac{|V|}{\Omega}\sum_{\mathbf{k}\neq\mathbf{p}} \langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}\rangle \tag{6}$$

We have (for $\mathbf{p} \neq \mathbf{k}$)

$$
\begin{aligned}
\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \rangle &= \langle 0 | \prod_{\boldsymbol{q}} (u_{\boldsymbol{q}} + v_{\boldsymbol{q}} c_{-\boldsymbol{q}\downarrow} c_{\boldsymbol{q}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \prod_{\boldsymbol{r}} (u_{\boldsymbol{r}} + v_{\boldsymbol{r}} c^\dagger_{\boldsymbol{r}\uparrow} c^\dagger_{-\boldsymbol{r}\downarrow}) | 0 \rangle \\
&= \langle 0 | \underbrace{\left(\prod_{\boldsymbol{q} \neq \boldsymbol{k}, \boldsymbol{p}} (u_{\boldsymbol{q}} + v_{\boldsymbol{q}} c_{-\boldsymbol{q}\downarrow} c_{\boldsymbol{q}\uparrow})(u_{\boldsymbol{q}} + v_{\boldsymbol{q}} c^\dagger_{\boldsymbol{q}\uparrow} c^\dagger_{-\boldsymbol{q}\downarrow}) \right)}_{\prod_{\boldsymbol{q}\neq \boldsymbol{k},\boldsymbol{p}} (u_{\boldsymbol{q}}^2 + v_{\boldsymbol{q}}^2) = 1} \\
&\quad \times (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow})(u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \\
&\quad \times c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})(u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow}) | 0 \rangle \\
&= \langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow})(u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \\
&\quad \times c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})(u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow}) | 0 \rangle
\end{aligned}
$$

More

$$
c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})(u_{\mathbf{p}} + v_{\mathbf{p}} c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow}) | 0 \rangle = v_{\mathbf{p}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) | 0 \rangle \tag{7}
$$

and

$$
\langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow})(u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} = \langle 0 | v_{\mathbf{k}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) \tag{8}
$$

from where

$$
\begin{aligned}
\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \rangle &= \langle 0 | v_{\mathbf{k}} (u_{\mathbf{p}} + v_{\mathbf{p}} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}) v_{\mathbf{p}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) | 0 \rangle \\
&= u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{p}} v_{\mathbf{p}}
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{|V|}{\Omega} \sum_{\mathbf{k} \neq \mathbf{p}} \langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} \rangle &= -\frac{|V|}{\Omega} \sum_{\mathbf{k} \neq \mathbf{p}} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{p}} v_{\mathbf{p}} \\
&\simeq -\frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(\sum_{\mathbf{p}} u_{\mathbf{p}} v_{\mathbf{p}} \right)
\end{aligned}
$$

So we finally get:

$$
\langle \Psi_{BCS} | H_{BCS} | \Psi_{BCS} \rangle = 2 \sum_{\mathbf{k}} \xi_{\mathbf{k}} v_{\mathbf{k}}^2 - \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(\sum_{\mathbf{p}} u_{\mathbf{p}} v_{\mathbf{p}} \right) \tag{9}
$$

(C.)

1. We have

$$
\begin{aligned}
\frac{\partial \langle H_{BCS} \rangle}{\partial v_{\mathbf{p}}} &= 2 \sum_{\mathbf{k}} \xi_{\mathbf{k}} \frac{\partial}{\partial v_{\mathbf{p}}} v_{\mathbf{k}}^2 - \frac{|V|}{\Omega} \frac{\partial}{\partial v_{\mathbf{p}}} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right)^2 \\
&= 4 \sum_{\mathbf{k}} \xi_{\mathbf{k}} v_{\mathbf{k}} \frac{\partial v_{\mathbf{k}}}{\partial v_{\mathbf{p}}} - 2 \frac{|V|}{\Omega} \left(\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \right) \left(\sum_{\mathbf{k}} \frac{\partial u_{\mathbf{k}}}{\partial v_{\mathbf{p}}} v_{\mathbf{k}} + \frac{\partial v_{\mathbf{k}}}{\partial v_{\mathbf{p}}} u_{\mathbf{k}} \right)
\end{aligned}
$$

More

$$
\frac{\partial v_{\mathbf{k}}}{\partial v_{\mathbf{p}}} = \delta_{k,p} \tag{10}
$$

and using $u_{\mathbf{k}}^2 = 1 - v_{\mathbf{k}}^2$

$$\begin{aligned}\frac{\partial u_{\mathbf{k}}}{\partial v_{\mathbf{p}}} &= \frac{\partial\sqrt{1-v_{\mathbf{k}}^2}}{\partial v_{\mathbf{p}}} \\ &= -\frac{v_{\mathbf{k}}}{\sqrt{1-v_{\mathbf{k}}^2}}\delta_{k,p} \\ &= -\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}\delta_{k,p}\end{aligned}$$

so we have

$$\frac{\partial\langle H_{BCS}\rangle}{\partial v_{\mathbf{p}}} = 4\xi_{\mathbf{p}}v_{\mathbf{p}} - 2\frac{|V|}{\Omega}\left(\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}}\right)\left(-\frac{v_{\mathbf{p}}^2}{u_{\mathbf{p}}} + u_{\mathbf{p}}\right) \tag{11}$$

and

$$\begin{aligned}\frac{\partial\langle H_{BCS}\rangle}{\partial v_{\mathbf{p}}} &= 0 \\ &\Leftrightarrow \\ 2\xi_{\mathbf{p}}v_{\mathbf{p}}u_{\mathbf{p}} &= \frac{|V|}{\Omega}\left(\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}}\right)\left(u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2\right)\end{aligned} \tag{12}$$

By defining

$$\Delta = \frac{|V|}{\Omega}\left(\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{13}$$

Eq.(12) becomes

$$2\xi_{\mathbf{p}}v_{\mathbf{p}}u_{\mathbf{p}} = \Delta\left(u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2\right) \tag{14}$$

2. We have the 2 equations

$$2\xi_{\mathbf{p}}v_{\mathbf{p}}u_{\mathbf{p}} = \Delta\left(u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2\right) \tag{15}$$

and

$$u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2 = 1 \tag{16}$$

to solve for the 2 unknowns $(u_{\mathbf{p}}, v_{\mathbf{p}})$. Of Eq.(16) we get $u_{\mathbf{p}} = \sqrt{1-v_{\mathbf{p}}^2}$ that we use in Eq.(15). We obtain:

$$\begin{aligned}\frac{2\xi_{\mathbf{p}}}{\Delta}\sqrt{1-v_{\mathbf{p}}^2}v_{\mathbf{p}} &= \left(1-2v_{\mathbf{p}}^2\right) \\ &\Rightarrow \\ \frac{4\xi_{\mathbf{p}}^2}{\Delta^2}(1-v_{\mathbf{p}}^2)v_{\mathbf{p}}^2 &= \left(1-2v_{\mathbf{p}}^2\right)^2 \\ &\Leftrightarrow \\ 4\frac{\Delta^2+\xi_{\mathbf{p}}^2}{\Delta^2}v_{\mathbf{p}}^4 - 4\frac{\Delta^2+\xi_{\mathbf{p}}^2}{\Delta^2}v_{\mathbf{p}}^2 + 1 &= 0\end{aligned}$$

We put $E_{\mathbf{p}}^2 = \Delta^2 + \xi_{\mathbf{p}}^2$ and we solve in $v_{\mathbf{p}}^2$:

$$\begin{aligned}v_{\mathbf{p}}^2 &= \frac{1}{2}\left(1 \pm \sqrt{1-\frac{\Delta^2}{E_{\mathbf{p}}^2}}\right) \\ &= \frac{1}{2}\left(1 \pm \frac{|\xi_{\mathbf{p}}|}{E_{\mathbf{p}}}\right)\end{aligned}$$

Choose the solution for Fermi Sea for the free system ($V = 0 \Rightarrow \Delta = 0$):

$$v_{\mathbf{p}} = \begin{cases} 1 & \text{if} \quad p < k_F \\ 0 & \text{if} \quad p > k_F \end{cases} \tag{17}$$

So you have to choose

$$v_{\mathbf{p}}^2 = \begin{cases} \frac{1}{2}\left(1 + \frac{|\xi_{\mathbf{p}}|}{E_{\mathbf{p}}}\right) & \text{if} \quad p < k_F \\ \frac{1}{2}\left(1 - \frac{|\xi_{\mathbf{p}}|}{E_{\mathbf{p}}}\right) & \text{if} \quad p > k_F \end{cases} \tag{18}$$

Noticing that $\xi_{\mathbf{p}} < 0$ when $p < k_F$ and $\xi_{\mathbf{p}} > 0$ when $p > k_F$, this result can be to summarize

$$v_{\mathbf{p}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\right) \tag{19}$$

By injecting Eq.(19) into Eq.(16), we obtain

$$\begin{aligned} u_{\mathbf{p}}^2 &= 1 - v_{\mathbf{p}}^2 \\ &= \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\right) \end{aligned}$$

In summary

$$\begin{cases} u_{\mathbf{p}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\right) \\ v_{\mathbf{p}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\right) \\ E_{\mathbf{p}}^2 = \Delta^2 + \xi_{\mathbf{p}}^2 \end{cases} \tag{20}$$

3. Let's go back to the definition of the gap Eq.(13) with the results Eq.(20) and $E_{\mathbf{k}}^2 = \Delta^2 + \xi_{\mathbf{k}}^2$:

$$\begin{aligned} \Delta &= \frac{|V|}{\Omega}\sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} \\ &= \frac{|V|}{2\Omega}\sum_{\mathbf{k}} \sqrt{\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)} \\ &= \frac{|V|}{2\Omega}\sum_{\mathbf{k}} \sqrt{\left(1 - \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2}\right)} \\ &= \frac{\Delta|V|}{2\Omega}\sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} \end{aligned}$$

In addition to the trivial solution $\Delta = 0$ which corresponds to the normal state, this equation has a solution $\Delta \neq 0$ (superconducting state), which is solution of the equation of the gap:

$$1 = \frac{|V|}{2\Omega}\sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} \tag{21}$$