The solution of the first two exercises are in the course notes.

(A.) Solving second quantized Hamiltonians.

We consider a system of N indistinguishable particles (bosons or fermions) with the Hamiltonian

$$H = \sum_{i,j} h_{ij} a_i^\dagger a_j.$$

1) For H to be Hermitian, $H^\dagger = H$. Since $(a_i^\dagger a_j)^\dagger = a_j^\dagger a_i$, we have:

$$H^\dagger = \sum_{i,j} h_{ij}^* (a_i^\dagger a_j)^\dagger = \sum_{i,j} h_{ij}^* a_j^\dagger a_i = \sum_{i,j} h_{ji}^* a_i^\dagger a_j.$$

Comparing with $H = \sum_{i,j} h_{ij} a_i^\dagger a_j$, we obtain:

$$h_{ji}^* = h_{ij} \quad \text{for all } i, j.$$

Thus, the matrix $h = (h_{ij})$ must be Hermitian: $h = h^\dagger$.

2) For bosons, the commutation relations are $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$, $[a_i^\dagger, a_j^\dagger] = 0$. For fermions, the anticommutation relations are $\{a_i, a_j^\dagger\} = \delta_{ij}$, $\{a_i, a_j\} = 0$, $\{a_i^\dagger, a_j^\dagger\} = 0$. We require that b_k satisfy the same (anti)commutation relations as a_j. For both cases, this implies that the transformation must be unitary. Indeed

$$[b_k, b_l^\dagger] = \sum_{i,j} u_{ki} u_{lj}^* [a_i, a_j^\dagger] = \sum_{i,j} u_{ki} u_{lj}^* \delta_{ij} = \sum_i u_{ki} u_{li}^* = (UU^\dagger)_{kl}.$$

Similarly, for fermions: $\{b_k, b_l^\dagger\} = (UU^\dagger)_{kl}$.

To have $[b_k, b_l^\dagger] = \delta_{kl}$ (bosons) or $\{b_k, b_l^\dagger\} = \delta_{kl}$ (fermions), we require:

$$UU^\dagger = I.$$

$[b_k, b_l] = 0$ or $\{b_k, b_l\} = 0$ are trivially always 0.

3) Since h is Hermitian, it can be diagonalized by a unitary matrix U

$$h = U \mathrm{diag}(e_1, e_2, \dots) U^\dagger,$$

where e_k are real eigenvalues.

Then:

$$H = \sum_{i,j} h_{ij} a_i^\dagger a_j = \sum_{i,j,k} u_{i,k} e_k u_{j,k}^* a_i^\dagger a_j = \sum_k e_k \left(\sum_i u_{i,k} a_i^\dagger \right) \left(\sum_i u_{j,k}^* a_j \right)$$

We can define the operators

$$b_k = \sum_j u_{j,k}^* a_j,$$

such that

$$H = \sum_k e_k b_k^\dagger b_k.$$

The operators b_k satisfy the same (anti)commutation relations as a_j (since U is unitary), and thus represent particles with the same statistics.

4) The Hamiltonian is:

$$H = \sum_k e_k b_k^\dagger b_k.$$

The number operators $\hat{n}_k = b_k^\dagger b_k$ have eigenvalues $n_k = 0, 1, 2, \ldots$ for bosons and $n_k = 0, 1$ for fermions. The spectrum is therefore $E = \sum_k e_k n_k$ for all combinations of n_k.

The ground state is the vacuum for all b_k if all $e_k \geq 0$:

$$|\mathrm{GS}\rangle = |0\rangle_b, \quad E_{\mathrm{GS}} = 0.$$

If some $e_k < 0$, for fermions the ground state is filled for all $e_k < 0$ and empty for $e_k > 0$:

$$E_{\mathrm{GS}} = \sum_{k: e_k < 0} e_k.$$

For bosons, if any $e_k < 0$, the energy is unbounded (since n_k can be arbitrarily large), implying an instability. The Hamiltonian is not bounded from below, and the system is not well-defined.

5)Particle-hole symmetry transforms $a \to a^\dagger$ (up to a phase). Define:

$$b = e^{i\phi} a^\dagger.$$

For fermions, this is a canonical transformation: $\{a, a^\dagger\} = 1$ implies $\{b, b^\dagger\} = 1$ (if properly normalized). It maps the vacuum to a filled state.

For bosons, $[a, a^\dagger] = 1$, but $[b, b^\dagger] = [a^\dagger, a] = -1$, which violates the bosonic commutation relation. Thus, it is not a canonical transformation for bosons. While similar transformations are possible to remedy to a negative e_k, the mapping is mathematically tricky as the Hilbert space is infinite.

6) Bonus: General quadratic Hamiltonian. Consider:

$$H = \sum_{i,j} \left(h_{ij} a_i^\dagger a_j + \frac{1}{2} M_{ij} a_i^\dagger a_j^\dagger + \frac{1}{2} N_{ij} a_i a_j \right).$$

Hermiticity requires:

$$H^\dagger = H \implies h_{ji}^* = h_{ij}, \quad M_{ji}^* = N_{ij}, \quad N_{ji}^* = M_{ij}.$$

We seek a Bogoliubov transformation:

$$b_k = \sum_j \left(u_{kj} a_j + v_{kj} a_j^\dagger \right).$$

To preserve the (anti)commutation relations, the matrices u and v must satisfy

- For bosons: $uu^\dagger - vv^\dagger = I$, $uv^T = vu^T$ (Symplectic matrices).
- For fermions: $uu^\dagger + vv^\dagger = I$, $uv^T + vu^T = 0$.

Now solving the Hamiltonian requires care. The Hamiltonian can be written in matrix form:

$$H = \frac{1}{2} \begin{pmatrix} \mathbf{a}^\dagger & \mathbf{a} \end{pmatrix} \mathcal{H} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} \mp \frac{1}{2} \mathrm{Tr}(h),$$

where $\mathbf{a} = (a_1, a_2, \dots)^T$, and the Bogoliubov-de Gennes (BdG) matrix is:

$$\mathcal{H} = \begin{pmatrix} h & M \\ M^\dagger & \pm h^\dagger \end{pmatrix}.$$

The top sign is for bosons, the bottom sign for fermions. $\mathcal{H}$ is Hermitian.

For fermions, we can simply diagonalize $\mathcal{H}$, which has a symmetric degenerate spectrum $\pm\varepsilon_k$. We can verify that ε_k and $-\varepsilon_k$ correspond to particle-hole conjugate particles, and therefore to the same excitations. Avoiding double counting, we obtain

$$H = \sum_k \epsilon_k (b_k^\dagger b_k - \frac{1}{2}) + \frac{1}{2} \mathrm{Tr}(h),$$

Taking ε_k positive, the groundstate is the vacuum of the b operators. In particular, the a particles will have (generally) fractional occupations.

For bosons, we have to take into account that we need a symplectic transformation. Instead of diagonalizing $\mathcal{H}$, we need to diagonalize

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{H}.$$

The rest of the computation follows as in the fermionic case.