(A.) The de Haas-van Alphen effect: The de Haas-van Alphen effect is an oscillatory variation of the diamagnetic susceptibility as a function of the magnetic field strength (B). The method provides details of the extremal areas of a Fermi surface. In 1930, de Haas and van Alphen measured the magnetization M of the semimetal bismuth (Bi) as a function of B. They observed that the magnetic susceptibility M/B is a periodic function of $1/B$. This phenomenon is typically observed at low temperatures and high magnetic fields in metals that satisfy $k_B T \lesssim \hbar\omega_c \ll \mu$, where μ is the chemical potential.

The energy of free electrons in a strong magnetic field is given by

$$E_{n,k_z,\pm} = \hbar\omega_c(n + \frac{1}{2}) + \frac{\hbar^2 k_z^2}{2m} \pm \frac{1}{2} g\mu_B B \qquad n = 0, 1, 2, \ldots,$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. The last term is the coupling between the spin of the electrons and the magnetic field (Zeeman effect), $g = 2$ is the Landé factor and $\mu_B = \frac{\hbar e}{2mc}$ is the Bohr magneton. The terms can be rearranged as

$$E_{n,k_z} = \hbar\omega_c n + \frac{\hbar^2 k_z^2}{2m} \qquad n = 0, 1, \ldots,$$

where each value with $n \neq 0$ occurs twice and the value with $n = 0$ occurs only once.

1. Using the definition of the free energy, $F = -k_B T \log Z$, where Z is the partition function, show that
$$F = \hbar\omega_c \left[\frac{1}{2} f(\mu) + \sum_{n=1}^{\infty} f(\mu - \hbar\omega_c n) \right],$$
where
$$f(\epsilon) = -\frac{mV}{2\pi^2\hbar^2\beta} \int_{-\infty}^{\infty} dk_z \log\left[1 + e^{\beta\left(\epsilon - \frac{\hbar^2 k_z^2}{2m}\right)} \right] \tag{1}$$

2. Derive the Poissons's formula
$$\frac{1}{2} g(0) + \sum_{n=1}^{\infty} g(n) = \int_0^{\infty} g(x)dx + \sum_{n=1}^{\infty} 2\mathrm{Re} \int_0^{\infty} g(x) e^{2\pi i n x} dx \tag{2}$$
for a general function $g(x)$. In order to do so, use the following Fourier series
$$\sum_{m=-\infty}^{\infty} \delta(x - m) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x},$$
and $\int_0^{\infty} \delta(x) g(x) dx = \frac{1}{2} g(0)$. Here $\delta(x)$ is the Dirac delta function.

3. Using the Eq. (2) with $g(x) = f(\mu - \hbar\omega_c x)$ defined in Eq. (1), show that the free energy can be rewritten as
$$F = F_0 + F_1, \qquad F_0 = \hbar\omega_c \int_0^{\infty} dx f(\mu - \hbar\omega_c x), \qquad F_1 = \frac{mV}{\beta\pi^2\hbar^2} \mathrm{Re} \sum_{n=1}^{\infty} I_n.$$

Show that F_0 is independent of B (without explicitly doing the integral). What is the expression of I_n?

4. By integrating by parts twice, show that in the limit $k_B T \ll \mu$,

$$F_1 = \frac{(m\hbar\omega_c)^{3/2} V}{2\pi^2 \hbar^3 \beta} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi n\mu}{\hbar\omega_c} - \frac{\pi}{4}\right)}{n^{3/2} \sinh\left(\frac{2\pi^2 n}{\hbar\omega_c \beta}\right)}. \tag{3}$$

In order to do so, you may need the change of variable $\xi = \beta(\hbar\omega_c x + \frac{\hbar^2 k_z^2}{2m} - \mu)$ and the following integrals:

$$\int_{-\infty}^{\infty} e^{-i\alpha k_z^2} dk_z = e^{-\frac{i\pi}{4}} \sqrt{\frac{\pi}{\alpha}},$$
$$\int_{-\infty}^{\infty} \frac{e^{\xi}}{(e^{\xi}+1)^2} e^{i\alpha\xi} d\xi = \frac{\pi\alpha}{\sinh(\pi\alpha)}.$$

5. Calculate the magnetic susceptibility

$$\frac{M}{B} = -\frac{1}{B}\frac{\partial}{\partial B}\left(\frac{F}{V}\right) \approx -\frac{m^{3/2}\mu\sqrt{\hbar\omega_c}}{B^2 \pi \hbar^3 \beta} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2\pi n\mu}{\hbar\omega_c} - \frac{\pi}{4}\right)}{\sqrt{n}\sinh\left(\frac{2\pi^2 n}{\hbar\omega_c\beta}\right)},$$

where M is the magnetisation density. We assume that only the most rapidly oscillating factors (the cosine term in Eq. (3)) needs to be differentiated.

6. What is the period of the oscillation?

7. Show that in the limit $\hbar\omega_c \ll k_B T$ (small magnetic field), the amplitude of the oscillation vanishes exponentially with $k_B T/(\hbar\omega_c)$.