(A.) Anti-ferromagnetic Holstein-Primakoff transformation: In the lecture, we introduced the Holstein-Primakoff transformation on a bipartite lattice for antiferromagnets. On the sublattice A, this is the same as the ferromagnetic case from session 3. On the sublattice B, the transformation writes:

$$\begin{cases} S_j^z = -S + b_j^\dagger b_j \\ S_j^+ = b_j^\dagger \left(\sqrt{2S - b_j^\dagger b_j}\right) \\ S_j^- = \left(\sqrt{2S - b_j^\dagger b_j}\right) b_j \end{cases} \tag{1}$$

Show that by imposing bosonic commutation relations $[b_j, b_i^\dagger] = \delta_{ij}$, $[b_i, b_j] = [b_j^\dagger, b_i^\dagger] = 0$, one recovers spin operators commutation relations.

(B.) Antiferromagnetic ground state: Consider the Antiferromagnetic Heisenberg model on a bipartite lattice, with $J > 0$:

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

The hypothesis that the quantum ground state is close to the Néel state allows us to apply spin wave theory in order to study the behavior of quantum fluctuations. By performing a Holstein-Primakoff transformation using the bosonic operators a_i, followed by a Fourier transformation and a Bogoliubov transformation with coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, we can write the Hamiltonian in diagonalform in terms of bosonic operators $\alpha_{\mathbf{k}}$ (see course):

$$H = E_0 + \sum_{\mathbf{k}} \omega_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}$$

Show that the ground state is given by:

$$|GS\rangle = {\prod_{\mathbf{k}}}' \frac{1}{u_{\mathbf{k}}} \exp\left(-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger\right) |0\rangle$$

where $|0\rangle$ is the vaccuum state for the bosons $a_{\mathbf{k}}$ and ${\prod}'_{\mathbf{k}}$ stands for the product on half the wavevectors, such that only one of each pair $\mathbf{k}$ and $-\mathbf{k}$ is included.

(C.) Antiferromagnetic correction to the magnetization: Let us consider the antiferromagnetic Heisenberg model ($J > 0$):

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

on the cubic lattice (dimension $D=3$). By performing a Holstein-Primakoff transformation followed by a Bogoliubov transformation with coefficients

$$u_{\mathbf{k}}^2 = \frac{6JS+\omega_{\mathbf{k}}}{2\omega_{\mathbf{k}}}, \quad v_{\mathbf{k}}^2 = \frac{6JS-\omega_{\mathbf{k}}}{2\omega_{\mathbf{k}}},$$

H H is written in terms of bosonic operators as (see course):

$$H = E_0 + \sum_k \omega_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}.$$

The dispersion relation is given by:

$$\omega_{\mathbf{k}} = 6JS\sqrt{1-\gamma_{\mathbf{k}}^2}, \quad \gamma_{\mathbf{k}} = \frac{1}{3}(\cos k_x + \cos k_y + \cos k_z)$$

and the staggered magnetization by:

$$M_{\text{alt}} \equiv \frac{1}{N}\left\langle \sum_{i\in A} S_i^z - \sum_{j\in B} S_j^z \right\rangle \simeq S + \delta M^{(1)} + \delta M^{(2)}.$$

where $\delta M^{(1)}$ et $\delta M^{(2)}$ are the corrections to the magnetization due, respectively, to quantum fluctuations ($T=0$) and thermal fluctuations ($T\neq 0$),

$$\begin{aligned} \delta M^{(1)} &= -\frac{1}{N}\sum_{\mathbf{k}} v_{\mathbf{k}}^2 \\ \delta M^{(2)} &= -\frac{1}{N}\sum_{\mathbf{k}} n_{\mathbf{k}}(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2). \end{aligned}$$

$n_{\mathbf{k}}$ is the Bose-Einstein distribution:

$$n_{\mathbf{k}} = \frac{1}{\exp\left(\frac{\omega_{\mathbf{k}}}{T}\right) - 1}$$

1. Show that in the first Brillouin zone ($-\pi < k_x, k_y, k_z \leq \pi$), the dispersion $\omega_{\mathbf{k}}$ vanishes for $\mathbf{k}=0$ and $\mathbf{k}=(\pi,\pi,\pi)$, as well as

$$\omega_{\mathbf{k}} = \omega_{\boldsymbol{q}+\mathbf{k}} \simeq 2\sqrt{3}JS|\mathbf{k}| \quad \text{for } \mathbf{k}\simeq 0 \text{ and } \boldsymbol{q}=(\pi,\pi,\pi)$$

2. Show that the correction to the magnetization $\delta M^{(2)}$ behaves as

$$\delta M^{(2)} \sim -\left(\frac{T}{JS}\right)^2 \quad \text{for } T \ll JS$$

3. The energy density of magnons is given by $E=\int \omega(\mathbf{k}) n_{\mathbf{k}} d^3k$. The specific heat is then defined as $c_{\text{magnetic}}^{\text{AF}} = \partial E/\partial T$ Show it behaves as

$$c_{\text{magnetic}}^{\text{AF}} \sim \left(\frac{T}{JS}\right)^3 \quad \text{for } T \ll JS$$

These results are to be compared to the ferromagnetic case. The difference in the T dependence is due to the fact that the dispersion of the antiferromagnetic model is linear around $\mathbf{k}=0$ while it is quadratic for the ferromagnetic model, as well as the presence of quantum fluctuations.