Coherence length of Cooper pairs. The purpose of this exercise is to determine the coherence length ℓ_c of the Cooper pairs. To this end, we use the expansion of the BCS state into states with a fixed number of particles (see course):

$$|\Phi_{BCS}\rangle = \sum_{N=0,2,4\cdots} |\Phi_N\rangle, \qquad \text{with} \qquad |\Phi_N\rangle = C\left(\sum_{\boldsymbol{k}} g_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}\right)^{N/2} |0\rangle \tag{1}$$

where $C = \frac{1}{(N/2)!}\left(\prod_{\boldsymbol{p}} u_{\boldsymbol{p}}\right)$, $g_{\boldsymbol{k}} = \frac{v_{\boldsymbol{k}}}{u_{\boldsymbol{k}}}$, $u^2_{\boldsymbol{k}} = \frac{1}{2}\left(1 + \frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$, $v^2_{\boldsymbol{k}} = \frac{1}{2}\left(1 - \frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$, and $E_{\boldsymbol{k}} = \sqrt{\xi^2_{\boldsymbol{k}} + \Delta^2}$.

1. Show that Φ_N can be decomposed into a product of two-particle wave functions

$$\langle \boldsymbol{r}_1\sigma_1, \cdots, \boldsymbol{r}_N\sigma_N|\Phi_N\rangle \sim \mathcal{A}\cdot\Big(\psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)\psi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\cdots\psi(\boldsymbol{r}_{N-1}\sigma_{N-1}, \boldsymbol{r}_N\sigma_N)\Big), \tag{2}$$

where

- $\psi(\boldsymbol{r}\sigma, \boldsymbol{r'}\sigma')$ is the two-particle wave function given by $\psi(\boldsymbol{r}\sigma, \boldsymbol{r'}\sigma') = \varphi(\boldsymbol{r} - \boldsymbol{r'})\eta_\uparrow(\sigma)\eta_\downarrow(\sigma')$,
- $\eta_{\sigma'}(\sigma) = \delta_{\sigma,\sigma'}$ is the spin wave function,
- $\varphi(\boldsymbol{r})$ is the spatial wave function defined by

$$\varphi(\boldsymbol{r}) = \frac{1}{\sqrt{\Omega}}\sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{3}$$

- $\mathcal{A}$ is the antisymmetrisation operator, defined by its action on a function $f(\boldsymbol{r}_1\sigma_1, \cdots, \boldsymbol{r}_N\sigma_N)$

$$\mathcal{A}\cdot f(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2, \cdots, \boldsymbol{r}_N\sigma_N) = \frac{1}{N!}\sum_{p\in S_N} s_p f(\boldsymbol{r}_{p(1)}\sigma_{p(1)}, \boldsymbol{r}_{p(2)}\sigma_{p(2)}, \cdots, \boldsymbol{r}_{p(N)}\sigma_{p(N)}), \tag{4}$$

- S_N is the set of permutations with N terms,
- and finally $s_p = (-1)^p = \pm 1$ is the signature of the permutation p.

Remember that

$$\begin{aligned} |\phi_1, \phi_2, \ldots, \phi_N\rangle &\equiv \frac{1}{\sqrt{N!}}\sum_{p\in S_N} \zeta^p |\phi_{p(1)}\rangle \otimes |\phi_{p(2)}\rangle \otimes \ldots \otimes |\phi_{p(n)}\rangle \\ &= c^\dagger_{\phi_1} c^\dagger_{\phi_2} \ldots c^\dagger_{\phi_N}|0\rangle \end{aligned} \tag{5}$$

where $\zeta = -1$ for fermions and $\zeta = +1$ for bosons. For example, in real-space representation,

$$|\boldsymbol{r}_1\sigma_1, \cdots, \boldsymbol{r}_N\sigma_N\rangle = \Psi^\dagger(\boldsymbol{r}_1\sigma_1)\cdots\Psi^\dagger(\boldsymbol{r}_N\sigma_N)|0\rangle$$

where Ψ is the field operator

$$\Psi^\dagger(\boldsymbol{r}\sigma) \equiv c^\dagger_{\boldsymbol{r}\sigma} = \frac{1}{\sqrt{\Omega}}\sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} c^\dagger_{\boldsymbol{k}\sigma}$$

such that

$$\langle \boldsymbol{r}\sigma|\boldsymbol{k}\sigma'\rangle = \frac{1}{\sqrt{\Omega}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta_{\sigma\sigma'} = \frac{1}{\sqrt{\Omega}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\eta_\sigma(\sigma') \tag{6}$$

2. Show that it is possible to express Eq. (2) in terms of antisymmetric wave functions:

$$\langle \boldsymbol{r}_1\sigma_1, \cdots, \boldsymbol{r}_N\sigma_N|\Phi_N\rangle \sim \mathcal{A}\cdot\Big(\psi_a(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)\psi_a(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\cdots\psi_a(\boldsymbol{r}_{N-1}\sigma_{N-1}, \boldsymbol{r}_N\sigma_N)\Big)$$

where the antisymmetric two-particle wave function ψ_a is defined by

$$\psi_a(\boldsymbol{r}\sigma, \boldsymbol{r}'\sigma') = \frac{1}{\sqrt{2}}\Big(\eta_\uparrow(\sigma)\eta_\downarrow(\sigma') - \eta_\downarrow(\sigma)\eta_\uparrow(\sigma')\Big)\ \varphi(\boldsymbol{r}-\boldsymbol{r}')$$

3. The coherence length ℓ_c of the pair is then calculated from the average radius ρ of the wave function ψ

$$\ell_c^2 \simeq \rho^2 = \frac{\langle\psi|\hat{R}^2|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int \varphi^*(\boldsymbol{R})R^2\varphi(\boldsymbol{R})d^3R}{\int \varphi^*(\boldsymbol{R})\varphi(\boldsymbol{R})d^3R} \tag{7}$$

Show that ρ^2 can be written as:

$$\rho^2 = \frac{\int d^3k(\boldsymbol{\nabla}_k g_{\boldsymbol{k}})^2}{\int d^3k g_{\boldsymbol{k}}^2} \tag{8}$$

and then as

$$\rho^2 = \frac{\left(\frac{d\xi_k}{dk}\right)^2_{\xi_k=0}\int_0^{\omega_D} d\xi_k\left(\frac{dg_k}{d\xi_k}\right)^2}{\int_0^{\omega_D} g_k^2 d\xi_k}. \tag{9}$$

Integrate, then using the weak-coupling approximation ($\Delta \ll \omega_D$) as well as

$$\left(\frac{d\xi_k}{dk}\right)_{\xi_k=0} = \hbar v_F \tag{10}$$

infer that

$$\ell_c \sim \frac{\hbar v_F}{\Delta}. \tag{11}$$

Usually, $\ell_c \sim 10^3$ Å, which should be compared with the lattice constant ∼1Å (∼ distance between the centers of mass of the pairs). Thereby, the spatial extent of the Cooper pairs is considerably larger than the distance between their centers of mass, and we cannot assume that they are independent.

Here are some useful integrals:

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} \qquad \text{et} \qquad \int \frac{x^2}{x^2+a^2}dx = x - a\arctan\frac{x}{a}$$

$$\int \frac{x}{\sqrt{x^2+a^2}}dx = \sqrt{x^2+a^2} \qquad \text{et} \qquad \int x\sqrt{x^2+a^2}dx = \frac{1}{3}(x^2+a^2)\sqrt{x^2+a^2}$$