In this series, we want to study the electron-pairing mechanism under the effect of an attractive two-body interaction $V(\boldsymbol{r}_1, \boldsymbol{r}_2)$,

$$H(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|). \tag{1}$$

This formulation is called the Cooper problem (L. N. Cooper, Phys. Rev. **104**, 1189 (1956)). To this end, we consider the wave function of vanishing total momentum $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0$:

$$\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \Psi(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)}. \tag{2}$$

In addition, we assume that Ψ is even, $\Psi(\boldsymbol{r}) = \Psi(-\boldsymbol{r})$, which implies that $\Psi_{\boldsymbol{k}} = \Psi_{-\boldsymbol{k}}$. As the two-electron state must be antisymmetric, this choice implies that the total spin must be zero ($S = 0$).

1. By setting

$$\epsilon_{\boldsymbol{k}} = \hbar^2 \boldsymbol{k}^2/2m \qquad \text{and} \qquad V_{\boldsymbol{k}-\boldsymbol{k}'} = \int d^3\boldsymbol{r} V(\boldsymbol{r}) e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}},$$

write the Schrödinger equation $H\Psi = E\Psi$ in the momentum representation. It is recalled that

$$\frac{1}{\Omega}\int d^3\boldsymbol{r} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k},\boldsymbol{0}}.$$

2. Assuming that the interaction does not depend on the momentum and is attractive, that is, $V_{\boldsymbol{k}-\boldsymbol{k}'} = -|v|$, show that the dispersion relation can be written in the form

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}'} \frac{1}{E - 2\epsilon_{\boldsymbol{k}'}} = -\frac{1}{|v|}. \tag{3}$$

3. We now assume that

   - both electrons are in the presence of a Fermi sea,
   - the attractive interaction comes from the effective electron-electron interaction due to phonons (see course). For simplicity, we assume that the interaction is attractive and independent of momentum ($-|v|$) if $\epsilon_F \leq \epsilon_{\boldsymbol{k}}, \epsilon_{\boldsymbol{k}'} \leq \epsilon_F + \hbar\omega_D$ and that it vanishes otherwise. $\hbar\omega_D$ is a characteristic energy of the phonons, which corresponds in general to the Debye frequency ω_D.

   Writing the energy of the two electrons as $E = 2\epsilon_F - \epsilon_b$, where $2\epsilon_F$ is the energy of the electrons without interaction and $\epsilon_b \geq 0$ is the binding energy of the electrons, calculate the binding energy ϵ_b for a weak interaction (in the sense that $\epsilon_b \ll \epsilon_F$) and with $\hbar\omega_D \ll \epsilon_F$ (always true for metals).

   It is recalled that for $x > b > 0$,

$$\int dx \frac{x^2}{(x^2 - b^2)} = x + \frac{b}{2}\ln\left|\frac{x-b}{x+b}\right|. \tag{4}$$

4. Using the density of states per spin at the Fermi level $\rho_F = mk_F/(2\pi^2\hbar^2)$, show that in the $\rho_F|v| \ll 1$ limit,

$$\epsilon_b \simeq 2(\hbar\omega_D)e^{-\frac{2}{\rho_F|v|}} \tag{5}$$

and that in the $\rho_F|v| \gg 1$ limit (strong coupling)

$$\epsilon_b \simeq (\hbar\omega_D)\rho_F|v|. \tag{6}$$