Exercise session 0

This series consists in demonstrating the core results of second quantization. The course appendix gives an introduction to these concepts.

(A.) Second quantization for bosons.

We consider a system of N bosons that can be a in k different states. We note N_i the number of particles in state i, with $\sum_i N_i = N$. In order to obtain a symmetric wavefunction, we have to sum over all permutations such that there are always N_i particle in state i, called permutation with repetition.

1) Show by recurrence that the number of permutation with repetition is

$$\text{Norm} = \frac{N!}{N_1!N_2!\dots N_k!} \tag{1}$$

We define the creation and annihilation operators $a_i^\dagger$ and a_i such that

$$a_i^\dagger |N_1, \dots N_i, \dots\rangle = \sqrt{N_i + 1}\, |N_1, \dots N_i + 1, \dots\rangle \tag{2}$$

$$a_i |N_1, \dots N_i, \dots\rangle = \sqrt{N_i}\, |N_1, \dots N_i - 1, \dots\rangle \tag{3}$$

with $a_i |N_1, \dots N_i, \dots\rangle = 0$ if $N_i = 0$.

2) Show that those operator fulfill the bosonic commutation relations

$$[a_i, a_j] = 0 \tag{4}$$

$$[a_i^\dagger, a_j^\dagger] = 0 \tag{5}$$

$$[a_i, a_j^\dagger] = \delta_{i,j} \tag{6}$$

3) Show that any one-body operator $\hat{F}^1$ can be rewritten in second quantization as

$$\hat{F}^1 = \sum_{ij} f_{ij} a_i^\dagger a_j \tag{7}$$

(B.) Second quantization for fermions.

We now turn to fermions.

1) Explain the normalization factor $1/\sqrt{N!}$.

2) Show the fermionic anti-commutation relations

$$\{c_i, c_j\} = 0 \tag{8}$$

$$\{c_i^\dagger, c_j^\dagger\} = 0 \tag{9}$$

$$\{c_i^\dagger, c_j\} = \delta_{i,j} \tag{10}$$

(C.) Solving second quantized Hamiltonians.

We consider a system of N indistinguishable particles - either bosons or fermions. We consider a generic Hamiltonian of the form

$$H = \sum_{i,j} h_{i,j} a_i^\dagger a_j$$

1) What is the constraints on h_{ij} due to Hermiticity

2) We consider a complex matrix U and the operators

$$b_k = \sum_j u_{k,j} a_j$$

What conditions U must satisfy for b_k to verify the same commutation relations as a_j?

3) Deduce that one can rewrite the Hamiltonian H as

$$H = \sum_k e_k b_k^\dagger b_k,$$

where b are particles with the same statistics as a. What is the relation between e and the matrix h?

4) What is the spectrum of the Hamiltonian H? What is the groundstate and its energy? For bosons, what happens if one of the e_k is negative?

5) A particle-hole symmetry invert the role of a and $a^\dagger$, such that we define an operator $b \propto a$. What are the condition on this transformation? Why is it problematic for bosons?

6) Bonus: question 1 to 4 for the Hamiltonian

$$H = \sum_{i,j} h_{i,j} a_i^\dagger a_j + M_{i,j} a_i^\dagger a_j^\dagger + N_{i,j} a_i a_j.$$

We will look for transformations

$$b_k = \sum_j u_{k,j} a_j + v_{k,j} a_j^\dagger.$$