Solid state physics III — **Written exam** — 17 January 2022

Professor: F. Mila.

Duration: 3 hours. Without documentation.

NB: The two problems are independent.

Problem 1: Bosonic and fermionic approaches to the spin-1/2 Heisenberg model

In this problem, we want to develop bosonic and fermionic approaches to the spin-1/2 antiferromagnetic Heisenberg model defined by the Hamiltonian:

$$\mathcal{H} = J \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j, \quad J > 0, \tag{1}$$

where the sum over $< i, j >$ denotes the sum over pairs of nearest neighbours. We will concentrate on the square lattice, we will denote the number of sites by N, we will assume periodic boundary conditions, and we will set $\hbar = 1$ throughout this problem.

Preliminary question

Starting from the commutation relation of spin operators

$$[S^x, S^y] = iS^z$$

(and similar relations after cyclic permutations), derive the commutation relations between S^+, S^- and S^z. What is the value of $\vec{S}^2$ for a spin 1/2?

Bosonic approach

1) Prove that a spin-1/2 at site i can be represented in terms of bosons $a_i^\dagger$, a_i as:

$$\begin{cases} S_i^z = \frac{1}{2} - a_i^+ a_i \\ S_i^+ = \sqrt{1 - a_i^+ a_i}\; a_i \\ S_i^- = a_i^+ \sqrt{1 - a_i^+ a_i} \end{cases}$$

2) For an antiferromagnet, if we use this representation on sublattice A, which representation should one use on sublattice B? Justify your answer.

3) Write the Hamiltonian of Eq. (1) in terms of the boson operators $a_i^\dagger$ and a_i keeping only terms involving at most two bosons.
4) Rewrite this Hamiltonian in terms of the Fourier transforms of these operators.
5) Using a Bogoliubov transformation, determine the spectrum of this Hamiltonian, and comment on the form of the dispersion.

Fermionic approach

1) Prove that a spin-1/2 operator at site i can be represented with two fermionic operators $(c_{i\uparrow}^\dagger, c_{i\uparrow})$ and $(c_{i\downarrow}^\dagger, c_{i\downarrow})$ as

$$\begin{cases} S_i^z = \frac{1}{2}(c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow}) \\ S_i^+ = c_{i\uparrow}^\dagger c_{i\downarrow} \\ S_i^- = c_{i\downarrow}^\dagger c_{i\uparrow} \end{cases}$$

with the constraint $c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow} = 1$.

2) One defines the bond operator on two sites by

$$\chi_{i,j} = c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow}$$

(a) Prove that

$$\chi_{i,j}^\dagger \chi_{i,j} = -2\vec{S}_i.\vec{S}_j + \frac{1}{2} \tag{2}$$

(b) Write the Hamiltonian of Eq. (1) in terms of $\chi_{i,j}^\dagger \chi_{i,j}$.

3) Write the mean-field version of this Hamiltonian obtained by decoupling the products $\chi_{i,j}^\dagger \chi_{i,j}$ with the help of the expectation values $\langle \chi_{i,j}^\dagger \rangle$ and $\langle \chi_{i,j} \rangle$.

NB: From now on, we will drop the contant from the mean-field Hamiltonian.

4) One looks for mean-field solutions such that, for any site $\vec{r}_i$ of sublattice A, the expectation values of the bond operators between this site and its nearest neighbours are given in terms of 4 complex numbers χ_1, χ_2, χ_3 and χ_4 by:

$$\langle \chi_{\vec{r}_i, \vec{r}_i + a\hat{x}} \rangle = \chi_1 \tag{3}$$
$$\langle \chi_{\vec{r}_i - a\hat{y}, \vec{r}_i} \rangle = \chi_2 \tag{4}$$
$$\langle \chi_{\vec{r}_i, \vec{r}_i - a\hat{x}} \rangle = \chi_3 \tag{5}$$
$$\langle \chi_{\vec{r}_i + a\hat{y}, \vec{r}_i} \rangle = \chi_4 \tag{6}$$

where a is the lattice parameter and $\hat{x}$ and $\hat{y}$ the unit vectors of the lattice.

(a) Justify briefly why the resulting mean-field Hamiltonian is periodic with 2 sites per unit cell.

(b) Let us choose $\vec{R}_1 = a\hat{x} + a\hat{y}$ and $\vec{R}_2 = -a\hat{x} + a\hat{y}$ as the basis vectors of this lattice, and let us denote by $\vec{r}$ the position of a unit cell made of a site of sublattice A at position $\vec{r} - \frac{a}{2}\hat{x}$ and a site of sublattice B at position $\vec{r} + \frac{a}{2}\hat{x}$ (see Fig. 1). Let us further simplify the notation and write $c_{\vec{r}-\frac{a}{2}\hat{x}\sigma}^\dagger = a_{\vec{r}\sigma}^\dagger$, $c_{\vec{r}+\frac{a}{2}\hat{x}\sigma}^\dagger = b_{\vec{r}\sigma}^\dagger$, $c_{\vec{r}-\frac{a}{2}\hat{x}\sigma} = a_{\vec{r}\sigma}$ and $c_{\vec{r}+\frac{a}{2}\hat{x}\sigma} = b_{\vec{r}\sigma}$, where $\sigma = \uparrow, \downarrow$ is the spin index. Like the original $c_{i\sigma}^\dagger, c_{i\sigma}$ fermions, $a_{\vec{r}\sigma}^\dagger, a_{\vec{r}\sigma}$ and $b_{\vec{r}\sigma}^\dagger, b_{\vec{r}\sigma}$ are of course fermionic operators.
Determine the values of $\vec{s}$ in terms of $\vec{r}$, $\vec{R}_1$ and $\vec{R}_2$ such that $b_{\vec{s}\sigma}$ is coupled to $a_{\vec{r}\sigma}^\dagger$ in the mean-field Hamiltonian.

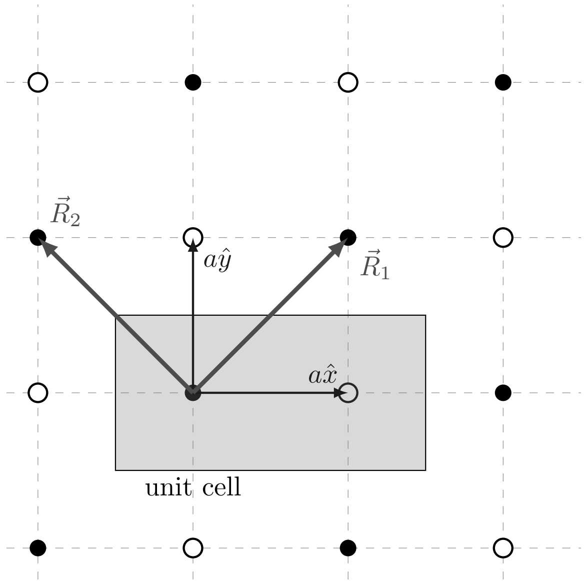

FIGURE 1 – Sketch of the lattice and of the two-site unit cell of question 4(b). Solid circles stand for sublattice A, open circles for sublattice B.

(c) For these values of $\vec{s}$, determine which of the mean-field parameters χ_i or χ_i^*, $i = 1, ..., 4$, multiplies the term $a^\dagger_{\vec{r}\sigma} b_{\vec{s}\sigma}$ of the mean-field Hamiltonian.

(d) Write the mean-field Hamiltonian as a sum over $\vec{r}$ and σ in terms of χ_1, χ_2, χ_3, χ_4, χ_1^*, χ_2^*, χ_3^*, χ_4^*, and of the operators $a^\dagger$, a, $b^\dagger$, b at site $\vec{r}$ and at neighboring sites.

(e) Show that, in terms of the Fourier transforms of $a^\dagger_{\vec{r}\sigma}$, $a_{\vec{r}\sigma}$, $b^\dagger_{\vec{r}\sigma}$ and $b_{\vec{r}\sigma}$, and up to a constant, the Hamiltonian takes the form:

$$H = \sum_{\vec{q},\sigma} (F_{\vec{q}}\ a^\dagger_{\vec{q}\sigma} b_{\vec{q}\sigma} + F^*_{\vec{q}}\ b^\dagger_{\vec{q}\sigma} a_{\vec{q}\sigma}) \tag{7}$$

where $F_{\vec{q}}$ and $F^*_{\vec{q}}$ are functions of χ_1, χ_2, χ_3, χ_4, χ_1^*, χ_2^*, χ_3^* and χ_4^* to be determined.

(f) Diagonalize a two-site fermionic Hamiltonian of the form

$$H = F a^\dagger b + F^* b^\dagger a \tag{8}$$

with $F = |F|e^{i\theta}$.
Hint: Look for linear combinations of a and b that bring the Hamiltonian into diagonal form.

(g) Deduce from the previous question the excitation energies of the mean-field Hamiltonian. How many branches do we have?

(h) Suppose that $\chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi e^{i\frac{\pi}{4}}$. Determine the explicit form of the spectrum in terms of $q_1 \equiv \vec{q}.\vec{R}_1$ and $q_2 \equiv \vec{q}.\vec{R}_2$. Show that, at half-filling, the Fermi energy $E_F = 0$, i.e. the Fermi surface is determined by the points where the excitation energy vanishes, and solve this equation. What is the peculiarity of the resulting Fermi ≪ surface ≫? Justify why the ground state corresponds to half-filling.

Final remark: The resulting spectrum is known as a Dirac spectrum, and this calculation is the basis of the theory of algebraic spin liquids. It is not of direct relevance for the square lattice (the ground state is known to have long-range order), but it is believed to apply to other cases.

Problem 2: Fractional quantum Hall effect in a harmonic well

In this problem, we will investigate the effect of a simple harmonic trapping potential $V_{trap} = \frac{1}{2}v_t r^2$ on the Laughlin states. We consider spinless electrons in two dimensions.

1) In this question, we rederive the expression of the single-particle orbitals in the symmetric gauge without trapping potential.

   NB: Questions (a) to (e) can be done without specifying the gauge.

   (a) Write the general expression of the Hamiltonian of a particule of charge $-e$ in a uniform magnetic field.

   (b) We introduce the operators

$$\hat{\Pi}_{x/y} = \hat{p}_{x/y} + \frac{e}{c}A_{x/y}. \tag{9}$$

   Compute the commutator $[\hat{\Pi}_x, \hat{\Pi}_y]$. Show that we can write the Hamiltonian as

$$H_{\text{Landau}} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega_c^2\hat{Q}^2 \tag{10}$$

   with $[\hat{P}, \hat{Q}] = i\hbar$.

   (c) Show that the Hamiltonian can be rewritten as

$$H = \hbar\omega_c(a^\dagger a + \frac{1}{2}) \tag{11}$$

   with $a, a^\dagger$ bosonic operators verifying $[a, a^\dagger] = 1$.

   (d) We now introduce the operators

$$\hat{X} = \hat{x} - \frac{1}{m\omega_c}\hat{\Pi}_y \tag{12}$$

$$\hat{Y} = \hat{y} + \frac{1}{m\omega_c}\hat{\Pi}_x. \tag{13}$$

   Show that they commute with the Hamiltonian and compute the commutator $[\hat{X}, \hat{Y}]$.

   (e) Show that there exist $b, b^\dagger$ bosonic operators such that $[b, H_{\text{Landau}}] = [b^\dagger, H_{\text{Landau}}] = 0$. Deduce the general form of the eigenstates of H_{Landau} in terms of a and b, and give the corresponding energies.

   (f) We use the convention $z = x - iy$ and $\partial = (\partial_x + i\partial_y)/2$. Show that, in the symmetric gauge defined by

$$A_x = -\frac{By}{2}, \quad A_y = \frac{Bx}{2} \tag{14}$$

   one can rewrite the a and b operators as:

$$a = \sqrt{2}\left(\frac{z}{4l_B} + l_B\overline{\partial}\right) \tag{15}$$

$$b = \sqrt{2}\left(\frac{z^*}{4l_B} + l_B\partial\right) \tag{16}$$

   where l_B is a length scale to be determined.

   (g) Show that the eigenstates of H_{Landau} in the lowest Landau level can be written as

$$\Phi_{n=0,m} = \frac{1}{\sqrt{2\pi l_B^2 m!}}\left(\frac{z}{\sqrt{2}l_B}\right)^m e^{-|z|^2/4/l_B^2}. \tag{17}$$

   Hint: start from the state with $n = m = 0$ and use induction.

2) We now add the trapping potential $V_{trap} = \frac{1}{2} v_t r^2$.
   (a) We define the operators

$$\hat{\Pi}_x(\alpha) = \hat{p}_x - \frac{\alpha}{2}\hat{y} \tag{18}$$

$$\hat{\Pi}_y(\alpha) = \hat{p}_y + \frac{\alpha}{2}\hat{x} \tag{19}$$

$$\hat{X}(\alpha) = \hat{x} - \alpha^{-1}\hat{\Pi}_y(\alpha) \tag{20}$$

$$\hat{Y}(\alpha) = \hat{y} + \alpha^{-1}\hat{\Pi}_x(\alpha). \tag{21}$$

   Express $\hat{\Pi}_x^2$ as a function of $\hat{\Pi}_x^2(\alpha)$, $\hat{Y}(\alpha)^2$ and $\hat{y}^2$.
   (b) Show that, for a certain value of α to be determined, one can rewrite the Hamiltonian $H_{\text{Landau}} + V_{trap}$ in the form

$$\lambda_1 \hat{\Pi}_x(\alpha)^2 + \lambda_2 \hat{\Pi}_y(\alpha)^2 + \lambda_3 \hat{X}(\alpha)^2 + \lambda_4 \hat{Y}(\alpha)^2. \tag{22}$$

   Give the expression of λ_1, λ_2, λ_3 and λ_4.
   (c) By analogy with the case without a trapping potention, show that the Hamiltonian can be written as:

$$H_{\text{Landau}} + V_{trap} = \hbar\tilde{\omega}_c(a^\dagger a + \frac{1}{2}) + \hbar\omega_t(b^\dagger b + \frac{1}{2}) \tag{23}$$

   for frequencies $\tilde{\omega}_c$ and ω_t to be determined.
   (d) Show that the eigenstates of the first Landau level have the same form as without the trapping potential, but with the length scale l_B replaced by a new length scale to be determined. Give the corresponding eigenvalues in the lowest Landau level.
3) We now consider a system of N electrons in the previous trapping potential.
   (a) In the presence of the trapping potential, and for non-interacting electrons, under which condition can we construct the ground state without using states in the higher Landau levels?
   (b) Write the wavefunction $\Psi_{1/3}(z_1, ..., z_N)$ of the Laughlin state $\nu = \frac{1}{3}$. What is its total angular momentum? What is its energy in the presence of the trapping potential?
   (c) We now add as an interaction term the projector V_1 onto the subspace of relative angular momentum 1 (in units of $\hbar$). Justify why the Laughlin state is a ground state of this Hamiltonian.
   (d) We now study some excitations of the Laughlin states in the presence of the trapping potential. Under which condition on the polynomial P are the states

$$\Psi^P_{1/3}(z_1, ...) = P(z_1, ...)\Psi_{1/3}(z_1, ...) \tag{24}$$

   also ground states of V_1?
   (e) Assuming that P satisfies the previous condition and is homogeneous of degree d_P, what is the energy of $\Psi^P_{1/3}$ in the presence of the trapping potential?
   (f) Give the first three excited energy levels, their degeneracy and the corresponding basis. Hint: do not try to orthonormalize the basis. Just assume that $\Psi^P_{1/3}(z_1, ...)$ and $\Psi^Q_{1/3}(z_1, ...)$ are linearly independent iff. P and Q are linearly independent.
   (g) Compare the energy spectrum obtained in the previous question with the first excited states of the toy Hamiltonian

$$H = v_F \sum_{k>0} k c_k^\dagger c_k, \tag{25}$$

   for a system of size L with $c^{(\dagger)}$ bosonic annihilation (creation) operators.

Final remark: This toy model describes chiral bosons (i.e. bosons that can only go in one direction). These excitations are typical of the edge states of topological phases such as the 1/3 plateau of the FQHE.