
SOLID STATE PHYSICS III **Written exam** January 31, 2020

Professor : F. Mila. Duration : 3 hours. Without documentation.

NB : The two exercises and the problem are independent.

Exercise 1

In this exercise, we consider the Hubbard model on two sites. It is defined by the Hamiltonian :

$$\mathcal{H} = -t \sum_{\sigma} \left(c^{\dagger}_{1\sigma} c_{2\sigma} + c^{\dagger}_{2\sigma} c_{1\sigma} \right) + U \sum_{i=1,2} n_{i\uparrow} n_{i\downarrow}$$

1. What is the total size of the Hilbert space, taking all possible numbers of particles into account ?
2. Find the eigenvalues of the Hamiltonian in the sectors with :
   (a) 0 particle.
   (b) 1 particle.
   (c) 2 particles.
   NB : This is the case treated in the lecture. It is longer than the other cases.
   (d) 3 particles.
   (e) 4 particles.
3. To check the results, it is useful to perform a particle-hole transformation.
   (a) Show that, if c and $c^{\dagger}$ are fermionic operators, the operators defined by $d^{\dagger} = c$ and $d = c^{\dagger}$ are also fermionic operators. NB : This is the definition of a particle-hole transformation.
   (b) Write the Hamiltonian in terms of the d-operators.
   (c) Show that this transformation allows one to relate the spectrum of sectors with different numbers of particles, and verify that these relations are satisfied by the results of question 2.

Exercise 2

In this exercise, we consider the spin-S antiferromagnetic Heisenberg model on the square lattice with interactions J_x and J_y along x and y respectively. Throughout one assumes that J_x and J_y are positive.

1. What is the classical ground state of this model ?

2. By performing appropriate Holstein-Primakoff transformations, show that up to terms of order $O(S)$ the Hamiltonian takes the form

$$\mathcal{H} = \sum_{\vec{k}} \left(B_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2} A_{\vec{k}} \left(a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} + a_{\vec{k}} a_{-\vec{k}} \right) \right) + C,$$

and determine $A_{\vec{k}}$, $B_{\vec{k}}$, and C as a function of J_x and J_y.

3. Justify briefly why this Hamiltonian can be put in diagonal form using operators defined by

$$\alpha_{\vec{k}} = u_{\vec{k}} a_{\vec{k}} + v_{\vec{k}} a^\dagger_{-\vec{k}}$$

and its hermitian conjugate. What is the condition on $u_{\vec{k}}$ and $v_{\vec{k}}$ for these operators to be bosonic?

For the rest of this exercise, we will assume without trying to prove it that, up to a constant, the Hamiltonian takes the form

$$\mathcal{H} = \sum_{\vec{k}} \omega_{\vec{k}} \alpha^\dagger_{\vec{k}} \alpha_{\vec{k}}$$

with

$$\omega_{\vec{k}} = \sqrt{B^2_{\vec{k}} - A^2_{\vec{k}}}$$

and that $u_{\vec{k}}$ and $v_{\vec{k}}$ are given by

$$u_{\vec{k}} = \sqrt{\frac{B_{\vec{k}} + \omega_{\vec{k}}}{2\omega_{\vec{k}}}}, \quad v_{\vec{k}} = \mathrm{sign}(A_{\vec{k}}) \sqrt{\frac{B_{\vec{k}} - \omega_{\vec{k}}}{2\omega_{\vec{k}}}}.$$

4. Show that the relevant order parameter at zero temperature is given by

$$m = S - \sum_{\vec{k}} v^2_{\vec{k}}.$$

What happens to this order parameter upon reducing J_y at J_x fixed?

Problem

The goal of this problem is to use an algebraic approach to derive the Landau levels of free electrons (with dispersion $E_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$) and of Dirac electrons with dispersion $E_{\vec{k}} = \pm v\hbar k$.

Part I

1. Recall the form of the Hamiltonian of a free particle of mass m and of charge $-e$ in the presence of a magnetic field $\vec{B}$.
2. From now on, we work in 2D, and we suppose that the field is uniform, $\vec{B} = B\hat{z}$, where $\hat{z}$ is the direction perpendicular to the plane of the motion.
   (a) Write the Hamiltonian in terms of the operators $\Pi_x = p_x + \frac{e}{c} A_x$ and $\Pi_y = p_y + \frac{e}{c} A_y$.
   (b) Prove that the Hamiltonian can be cast into the form of a harmonic oscillator, and determine its frequency.

(c) Express the creation and annihilation operators $a^\dagger$ and a of this harmonic oscillator as a function of Π_x and Π_y.

Part II

In this part, the goal is to derive the Hamiltonian of the Dirac equation in 2D in terms of a 2-component spinor.

1. Assuming that the Hamiltonian acts in a two-component Hilbert space, i.e.

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix}$$

and that its eigenvalues are of the form $E_{\vec{k}} = \pm v\hbar k$, what are the eigenvalues of $\mathcal{H}^2$?

2. Prove that the operator

$$\mathcal{H} = v \begin{pmatrix} 0 & p_y - ip_x \\ p_y + ip_x & 0 \end{pmatrix}$$

is such that $\mathcal{H}^2$ has the right spectrum.

3. Check that one can find eigenstates of $\mathcal{H}$ of eigenvalue $E_{\vec{k}} = v\hbar k$ of the form

$$\begin{pmatrix} e^{i\vec{k}.\vec{r}} \\ ie^{-i\varphi(\vec{k})}e^{i\vec{k}.\vec{r}} \end{pmatrix}$$

where $\varphi(\vec{k})$ is a phase factor to be determined.

4. Determine the form of the eigenstates of $\mathcal{H}$ with eigenvalue $E_{\vec{k}} = -v\hbar k$.

Part III

The goal of this part is to solve the problem of Dirac electrons in the presence of a uniform magnetic field $\vec{B} = B\hat{z}$, assuming that the Hamiltonian is still obtained by the substitution $\vec{p} \to \vec{\Pi}$ in the Hamiltonian (Peierls substitution).

1. Write the Hamiltonian $\mathcal{H}$ of Dirac electrons in magnetic field in terms of the creation and annihilation operators $a^\dagger$ and a of the question 2(c) of Part I.
2. Calculate the square of this Hamiltonian, $\mathcal{H}^2$.
3. Diagonalize $\mathcal{H}^2$, and specify the degeneracy of its eigenvalues.
   Hint : Look for eigenstates of the form

$$\begin{pmatrix} |p\rangle \\ \alpha|q\rangle \end{pmatrix}$$

   with $a^\dagger a|p\rangle = p|p\rangle$, where α is just a complex coefficient.
4. Using the result of the previous question, determine the spectrum of $\mathcal{H}$. Are the Landau levels equidistant ?

Note : This is a model for the Landau levels of graphene, whose band structure consists of two Dirac cones close to the Fermi energy.