Professor : F. Mila. Duration : 2 hours. Without document.

NB1 : The two problems are independent.

NB2 : Some useful formulae are given at the end.

Problem 1

In this problem, we study two examples of ferromagnetic spin chains.

Part I : In this part, we consider a spin-S chain described by the Hamiltonian

$$\mathcal{H} = -J \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1}, \quad J > 0,$$

with periodic boundary conditions ($\vec{S}_{N+1} \equiv \vec{S}_1$). The lattice spacing is denoted by a.

1. Show that the state $|m_1 = S, ..., m_N = S\rangle \equiv |S....S\rangle$ is a ground state, and calculate its energy. In this expression, m_i is the eigenvalue of S_i^z at site i.
2. Determine the degeneracy of the ground state.
3. Explain why one can diagonalize the Hamiltonian in the sector $S^z_{\text{tot}} = NS - 1$.
4. Diagonalize the Hamiltonian in this sector. Plot the dispersion $\epsilon(k)$, and discuss its form close to $k = 0$.

Part II : One now considers a spin chain with an additional next-nearest neighbour interaction described by the Hamiltonian

$$\mathcal{H} = -J_1 \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} - J_2 \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+2}, \quad J_1, J_2 > 0,$$

with periodic boundary conditions ($\vec{S}_{N+1} \equiv \vec{S}_1$ and $\vec{S}_{N+2} \equiv \vec{S}_2$).

1. Show that the state $|S....S\rangle$ is still a ground state, and calculate its energy. What is the degeneracy of the ground state ?
2. Diagonalize the Hamiltonian in the sector $S^z_{\text{tot}} = NS - 1$. Plot and discuss the dispersion (minima, maxima). What happens in the limit $J_1 \to 0$? Comment the result.

Problem 2

In this problem, we are going to work out a simple model of the magnetization plateaux in the Integer Quantum Hall Effect in two steps, first rederiving results of the lecture notes on the Landau levels in the Landau gauge in the presence of a uniform electric field.

Part I : In this part, we consider the problem of Landau levels for an electron of mass m and charge $-e$ moving in 2D in the (x, y) plane in a uniform magnetic field along z of intensity B in the Landau gauge : $A_x = -By$, $A_y = A_z = 0$. We also add a uniform electric field E along y, which corresponds to a potential $V(y) = eEy$.

1. Write down the Hamiltonian.
2. Justify why it can be diagonalized by a factorized wave-function of the form

$$\psi(x, y) = e^{ik_x x}\varphi(y)$$

3. Show that the problem in the y direction is equivalent to a 1D harmonic oscillator. Determine the center $y(k_x)$ of this harmonic oscillator.
4. Give the expression of the eigenenergies $E(n, k_x)$, where n is the quantum number of the harmonic oscillator and k_x the momentum along x, as a function of the cyclotron frequency $\omega_c = \frac{eB}{mc}$.
5. Calculate the expectation value of the current in the x direction $\hat{j}_x$ in the eigenstate $|n, k_x\rangle$.
   Hint : Remember that the current is given by $\vec{j} = -e\vec{v} = -\frac{e}{m}(\vec{p} + \frac{e}{c}\vec{A})$.

Part II : On top of the electric field E, we now add a confining potential $W(y) = \frac{1}{2}m\omega_0^2 y^2$, where ω_0 is a parameter not to be confused with the cyclotron frequency ω_c.

1. Write down the Hamiltonian.
2. Show that the problem in the y direction is still equivalent to a 1D harmonic oscillator. Determine the frequency ω and the center $y(k_x)$ of this harmonic oscillator.
3. Give the expression of the eigenenergies $E(n, k_x)$. Show that $E(n, k_x)$ has a minimum at a wave vector $k_{x,min}$.
4. Assuming that the system is limited by $-\frac{L_y}{2} \leq y(k_x) \leq \frac{L_y}{2}$, determine the allowed values of k_x.
5. Determine the condition on E and ω_0 for which $k_{x,min}$ lies in the range of allowed values.
6. Calculate the expectation value of the current in the x direction $\hat{j}_x$ in the eigenstate $|n, k_x\rangle$. Does it depend on k_x ? Calculate its value at $k_{x,min}$.
7. Explain why, upon filling the band $E(n, k_x)$ for a given n, the conductance remains flat for a while if the condition of question 5 is fulfilled.

Some useful formulae

$$S^+|m\rangle = \sqrt{S(S+1) - m(m+1)}\||m+1\rangle$$

$$S^-|m\rangle = \sqrt{S(S+1) - m(m-1)}\||m-1\rangle.$$