Quantum mechanics for non-physicists

PHYS-344 / **5 credits**

Review of **mathematics of single quantum** bit systems

Quantum state is an element of a **vector space**

Dirac notation

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\langle\Psi| = \langle 0|\alpha^* + \langle 1|\beta^*$$

Matrix notation

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \alpha\begin{pmatrix}1 \\ 0\end{pmatrix} + \beta\begin{pmatrix}0 \\ 1\end{pmatrix}$$

$$(\alpha^* \quad \beta^*) = \alpha^*(1 \quad 0) + \beta^*(0 \quad 1)$$

Inner product

$$\langle\Psi|\Psi'\rangle = (\alpha^* \beta^*) \times \begin{pmatrix}\alpha' \\ \beta'\end{pmatrix} = \alpha^*\alpha' + \beta^*\beta'$$

Basis states are **orthonormal**

$$\langle 0|1\rangle = \langle 1|0\rangle = 0$$

$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$

Normalization

$$\langle\Psi'|\Psi'\rangle = \langle\Psi|\Psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$

Outer product

$$|\Psi\rangle\langle\Psi'| = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \times (\alpha'^* \quad \beta'^*) = \begin{pmatrix}\alpha\alpha'^* & \alpha\beta'^* \\ \beta\alpha'^* & \beta\beta'^*\end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix}1 \\ 0\end{pmatrix}(1 \quad 0) = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix} \qquad |0\rangle\langle 1| = \begin{pmatrix}1 \\ 0\end{pmatrix}(0 \quad 1) = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix}0 \\ 1\end{pmatrix}(0 \quad 1) = \begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix} \qquad |1\rangle\langle 0| = \begin{pmatrix}0 \\ 1\end{pmatrix}(1 \quad 0) = \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}$$

Global phase is irrelevant

$$\alpha = |\alpha|\exp(i\phi_1)$$

$$\beta = |\beta|\exp(i\phi_2)$$

$$\phi = \phi_2 - \phi_1$$

These two state vectors are the same for all purposes

$$|\Psi\rangle = \exp(i\phi_1)\Big(|\alpha||0\rangle + |\beta|\exp(i\phi)|1\rangle\Big) \quad \Rightarrow \quad |\Psi\rangle = |\alpha||0\rangle + |\beta|\exp(i\phi)|1\rangle$$

but relative phase matters

Linear **operators** map a vector space on itself

Linear operator acts on a vector to produce a new vector

$$\hat{L}|\Psi\rangle = \alpha\hat{L}|0\rangle + \beta\hat{L}|1\rangle$$

$$\langle\Psi|\hat{L} = \langle 0|\hat{L}\alpha^* + \langle 1|\hat{L}\beta^*$$

$$(\hat{L}|\Psi\rangle)^\dagger = \langle\Psi|\hat{L}^\dagger$$

Operators are specified by acting on (any) **basis states**

$$\hat{L}|0\rangle = L_{00}|0\rangle + L_{01}|1\rangle, \quad \hat{L}|1\rangle = L_{10}|0\rangle + L_{11}|1\rangle$$

Matrix elements

$$L_{00} = \langle 0|\hat{L}|0\rangle \quad L_{01} = \langle 1|\hat{L}|0\rangle$$

$$L_{10} = \langle 0|\hat{L}|1\rangle \quad L_{11} = \langle 1|\hat{L}|1\rangle$$

$$\hat{L}|\Psi\rangle = \begin{pmatrix} L_{00} & L_{10} \\ L_{01} & L_{11} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\langle\Psi|\hat{L}^\dagger = (\alpha^* \quad \beta^*)\begin{pmatrix} L_{00}^* & L_{01}^* \\ L_{10}^* & L_{11}^* \end{pmatrix}$$

Hermitian operators *represent physical observables* through their **eigenvalues** and **eigenvectors**

$$\hat{L}^\dagger = \hat{L}$$

$$\hat{L}|0_L\rangle = \lambda_0|0_L\rangle$$

$$\hat{L}|1_L\rangle = \lambda_1|1_L\rangle$$

real $\lambda_0 = \lambda_0^*, \quad \lambda_1 = \lambda_1^*$

orthogonal $\langle 0_L|1_L\rangle = 0$

Eigenbasis $|0_L\rangle, |1_L\rangle$ $\quad \hat{L} = \lambda_0|0_L\rangle\langle 0_L| + \lambda_1|1_L\rangle\langle 1_L|$

Unitary operators rotate basis: $|0_L\rangle = \hat{U}|0\rangle, \quad |1_L\rangle = \hat{U}|1\rangle$

$$\hat{U}^{-1} = \hat{U}^\dagger$$

$$|0\rangle = \hat{U}^\dagger|0_L\rangle \quad |1\rangle = \hat{U}^\dagger|1_L\rangle$$

$$\hat{U}|\Psi\rangle = \begin{pmatrix} \langle 0|0_L\rangle & \langle 0|1_L\rangle \\ \langle 1|0_L\rangle & \langle 1|1_L\rangle \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\hat{L}|\Psi\rangle = \left[\hat{U}\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}\hat{U}^\dagger\right]\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.linalg as la
```

Dirac form $\hat{L} = L_{00}|0\rangle\langle 0| + L_{01}|1\rangle\langle 0| + L_{10}|0\rangle\langle 1| + L_{11}|1\rangle\langle 1|$ $\quad \hat{L} = \hat{U}\Big(\lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|\Big)\hat{U}^\dagger$

Pauli operators generate all unitary and hermitian operators

Computational basis

$$|\Psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\hat{Z}|\Psi\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\hat{X}|\Psi\rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\hat{Y}|\Psi\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$[\hat{X},\hat{Y}] = 2i\hat{Z}$$
$$[\hat{Z},\hat{X}] = 2i\hat{Y}$$
$$[\hat{Y},\hat{Z}] = 2i\hat{X}$$
$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{I}$$

prove it!

Eigenbasis

"**Computational**" basis $|0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$ $|1\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$

$$\hat{Z}|0\rangle = +1|0\rangle$$
$$\hat{Z}|1\rangle = -1|1\rangle$$

$$\hat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\hat{X}|0\rangle = |1\rangle$$
$$\hat{X}|1\rangle = |0\rangle$$
$$\hat{X}|+\rangle = +1|+\rangle$$
$$\hat{X}|-\rangle = -1|-\rangle$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$$

$$\hat{X} = |+\rangle\langle +| - |-\rangle\langle -|$$

$$\hat{Y}|0\rangle = i|1\rangle$$
$$\hat{Y}|1\rangle = -i|0\rangle$$
$$\hat{Y}|+i\rangle = +1|+i\rangle$$
$$\hat{Y}|-i\rangle = -1|-i\rangle$$

$$|\pm i\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm i\frac{1}{\sqrt{2}}|1\rangle$$

$$\hat{Y} = |+i\rangle\langle +i| - |-i\rangle\langle -i|$$

Changing between the basis

Global phase does not matter!

$|1\rangle - |0\rangle$ equivalent to $|0\rangle - |1\rangle$

$i|0\rangle + |1\rangle$ equivalent to $|0\rangle - i|1\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}(\hat{X}+\hat{Z})|0\rangle$$
$$|-\rangle = \frac{1}{\sqrt{2}}(\hat{X}+\hat{Z})|1\rangle$$
$$|+i\rangle = \frac{1}{\sqrt{2}}(\hat{Y}+\hat{Z})|0\rangle$$
$$|-i\rangle = \frac{1}{\sqrt{2}}(\hat{Y}+\hat{Z})|1\rangle$$

Any hermitian operator

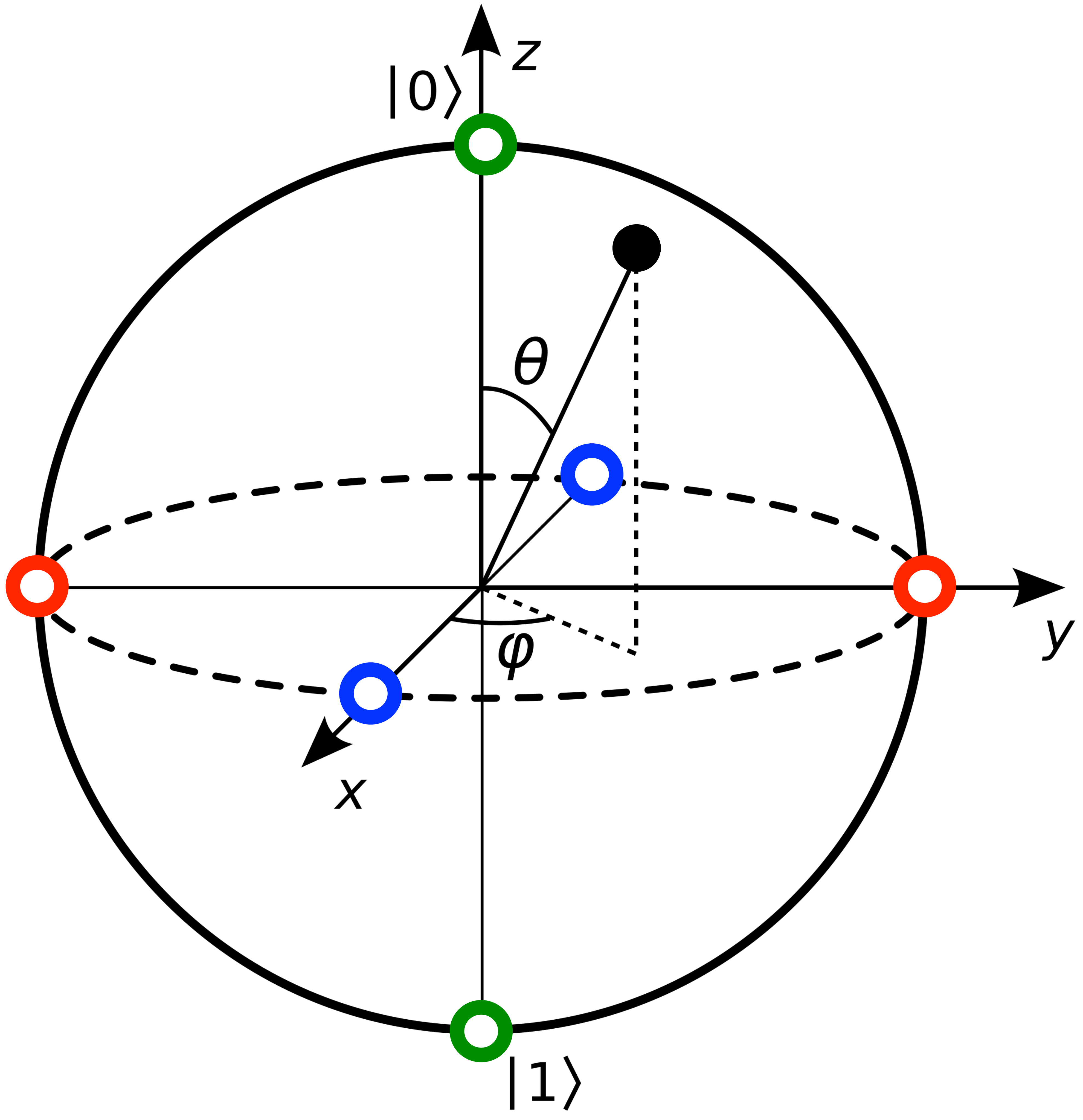
$$\hat{L} = \epsilon\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}$$

Any unitary operator

$$\hat{U} = \exp(i\delta\hat{L})$$

prove it!

Visualizing single qubit states on the **Bloch sphere**



$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + \exp(i\varphi)\sin\frac{\theta}{2}|1\rangle$$

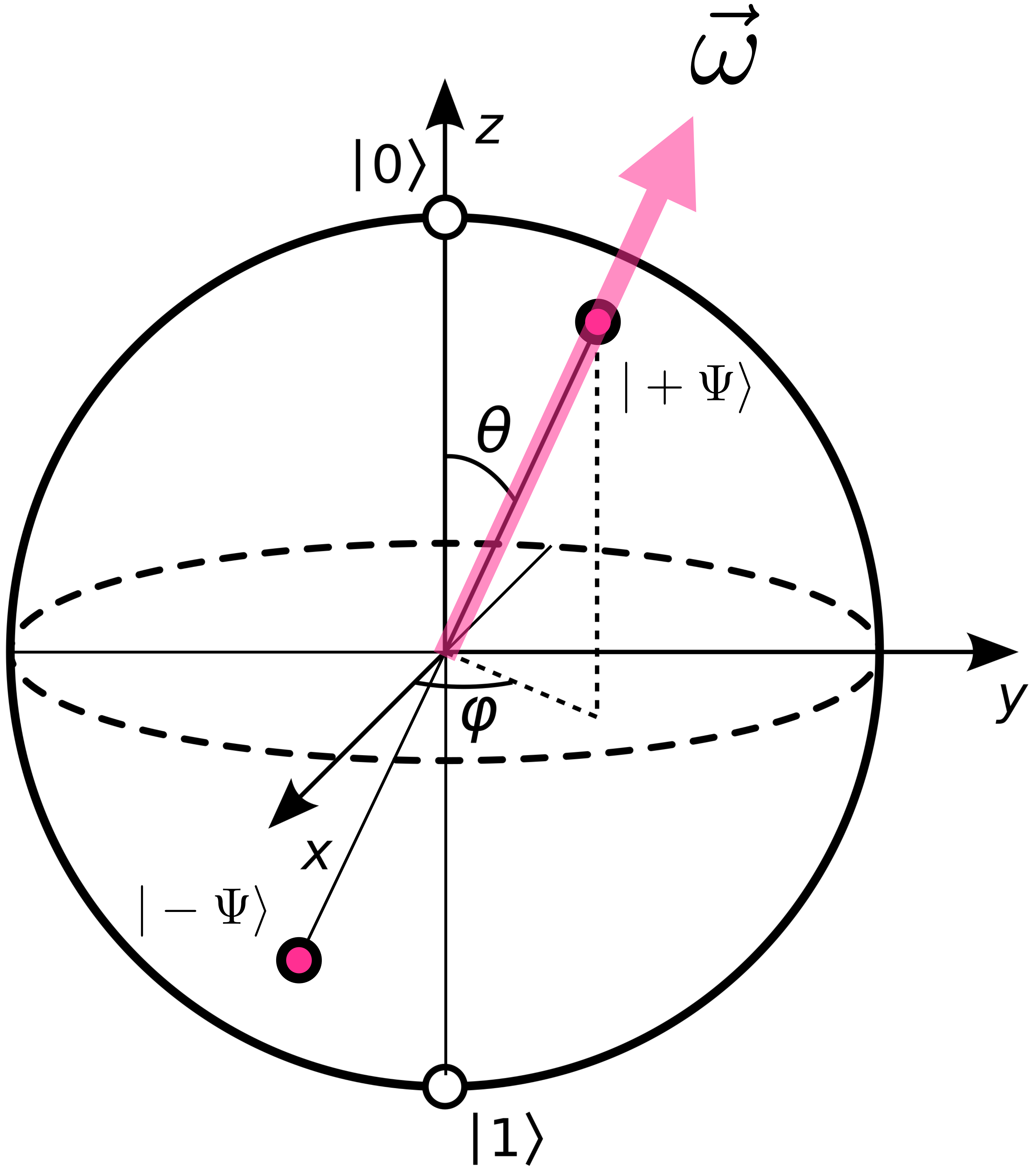
latitude θ, longitude φ

$|0\rangle$ north pole
$|1\rangle$ south pole
eigenstates of **Z**

$$|\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$$
eigenstates of **X**

$$|\pm i\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm i\frac{1}{\sqrt{2}}|1\rangle$$
eigenstates of **Y**

Geometric link between a **hermitian** operator and its **eigenstates**



Direction vector: $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$

$$\omega_z = \omega\cos\theta$$
$$\omega_x = \omega\sin\theta\cos\varphi \qquad \omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$
$$\omega_y = \omega\sin\theta\sin\varphi$$

Hermitian operator: $\hat{\Omega} = \omega_x\hat{X} + \omega_y\hat{Y} + \omega_z\hat{Z}$

$$\hat{\Omega} = \omega\begin{pmatrix}\cos\theta & \exp(-i\varphi)\sin\theta \\ \exp(i\varphi)\sin\theta & -\cos\theta\end{pmatrix}$$

$$\hat{\Omega}|+\Psi\rangle = +\omega|+\Psi\rangle$$

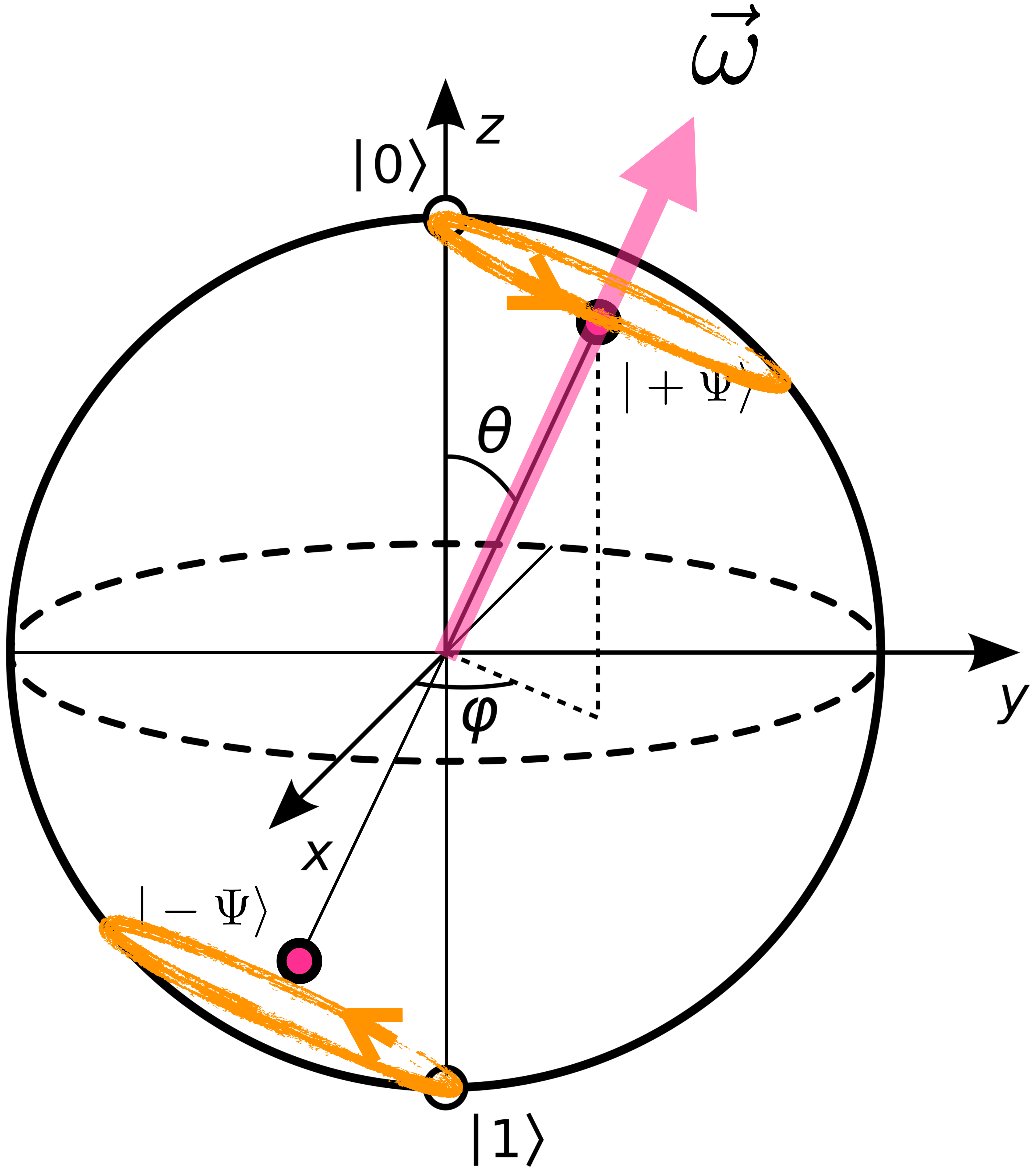
$$\hat{\Omega}|-\Psi\rangle = -\omega|-\Psi\rangle$$

Verify it!

$$|+\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + \exp(i\varphi)\sin\frac{\theta}{2}|1\rangle$$

$$|-\Psi\rangle = \sin\frac{\theta}{2}|0\rangle - \exp(i\varphi)\cos\frac{\theta}{2}|1\rangle$$

Unitary operators correspond to **rotations** in the Bloch sphere



General **unitary**: $\hat{R} = \exp(-i\hat{\Omega}t/2)$ $\quad \vec{\omega} = (\omega_x, \omega_y, \omega_z)$

$$\hat{\Omega} = \omega_x \hat{X} + \omega_y \hat{Y} + \omega_z \hat{Z} \qquad \omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

Eigenstates basis

$$\hat{\Omega}|\pm\Psi\rangle = \pm\omega|\pm\Psi\rangle$$

$$\hat{R}|+\Psi\rangle = \exp(-i\omega t/2)|+\Psi\rangle$$

$$\hat{R}|-\Psi\rangle = \exp(+i\omega t/2)|-\Psi\rangle$$

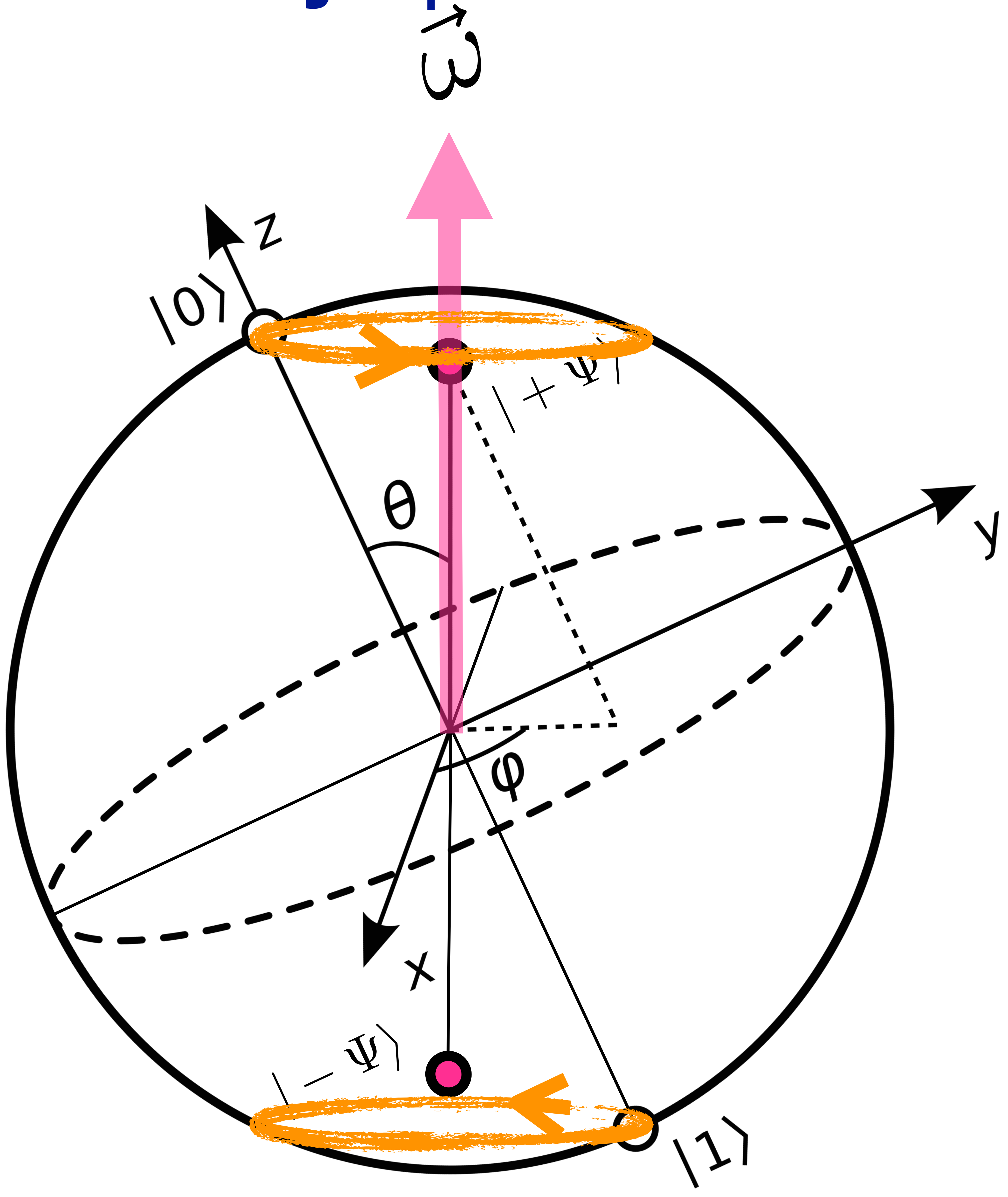
Computational basis

$$|0\rangle = \alpha_0|+\Psi\rangle + \beta_0|-\Psi\rangle \qquad \hat{R}|0\rangle = \alpha_0|+\Psi\rangle + \beta_0 \exp(+i\omega t)|-\Psi\rangle$$

$$|1\rangle = \beta_0^*|+\Psi\rangle - \alpha_0^*|-\Psi\rangle \qquad \hat{R}|1\rangle = \beta_0^*|+\Psi\rangle - \alpha_0^* \exp(+i\omega t)|-\Psi\rangle$$

$\hat{R}$ rotates any state by angle ωt around the axis of $\vec{\omega}$

Unitary operators correspond to **rotations** in the Bloch sphere



Basis change is like tilting your head

General **unitary**: $\hat{R} = \exp(-i\hat{\Omega}t/2)$ $\quad \vec{\omega} = (\omega_x, \omega_y, \omega_z)$

$$\hat{\Omega} = \omega_x \hat{X} + \omega_y \hat{Y} + \omega_z \hat{Z} \qquad \omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

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$$\hat{\Omega}|\pm\Psi\rangle = \pm\omega|\pm\Psi\rangle$$

$$\hat{R}|+\Psi\rangle = \exp(-i\omega t/2)|+\Psi\rangle$$

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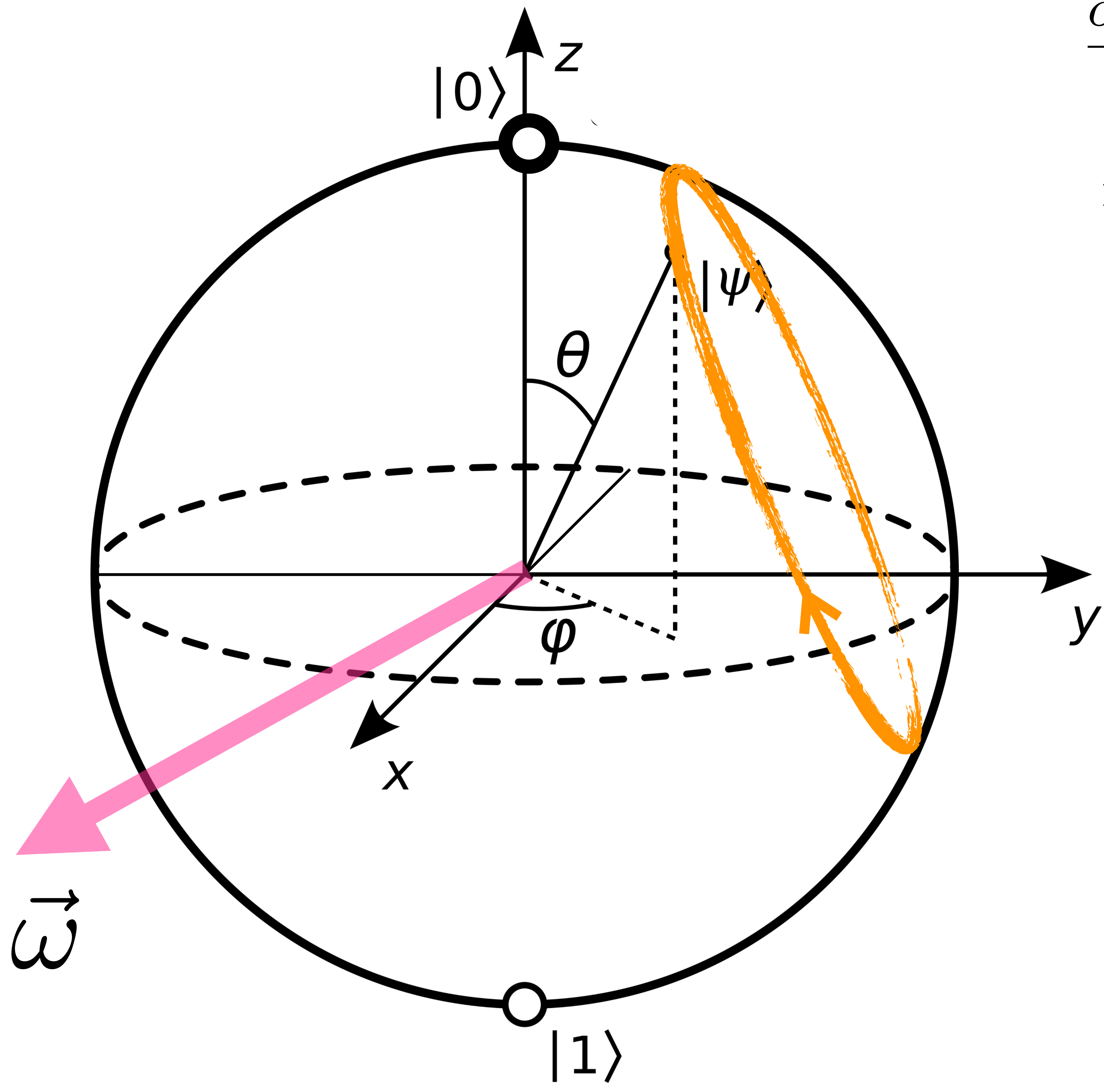
$$|0\rangle = \alpha_0|+\Psi\rangle + \beta_0|-\Psi\rangle \qquad \hat{R}|0\rangle = \alpha_0|+\Psi\rangle + \beta_0 \exp(+i\omega t)|-\Psi\rangle$$

$$|1\rangle = \beta_0^*|+\Psi\rangle - \alpha_0^*|-\Psi\rangle \qquad \hat{R}|1\rangle = \beta_0^*|+\Psi\rangle - \alpha_0^* \exp(+i\omega t)|-\Psi\rangle$$

$\hat{R}$ rotates any state by angle ωt around the axis of $\vec{\omega}$

Time-evolution of qubit states is set by the **Schrodinger** equation



$$\frac{\partial|\Psi\rangle}{\partial t} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle \qquad \hat{H}/\hbar = -\frac{\omega_x}{2}\hat{X} - \frac{\omega_Y}{2}\hat{Y} - \frac{\omega_z}{2}\hat{Z}$$

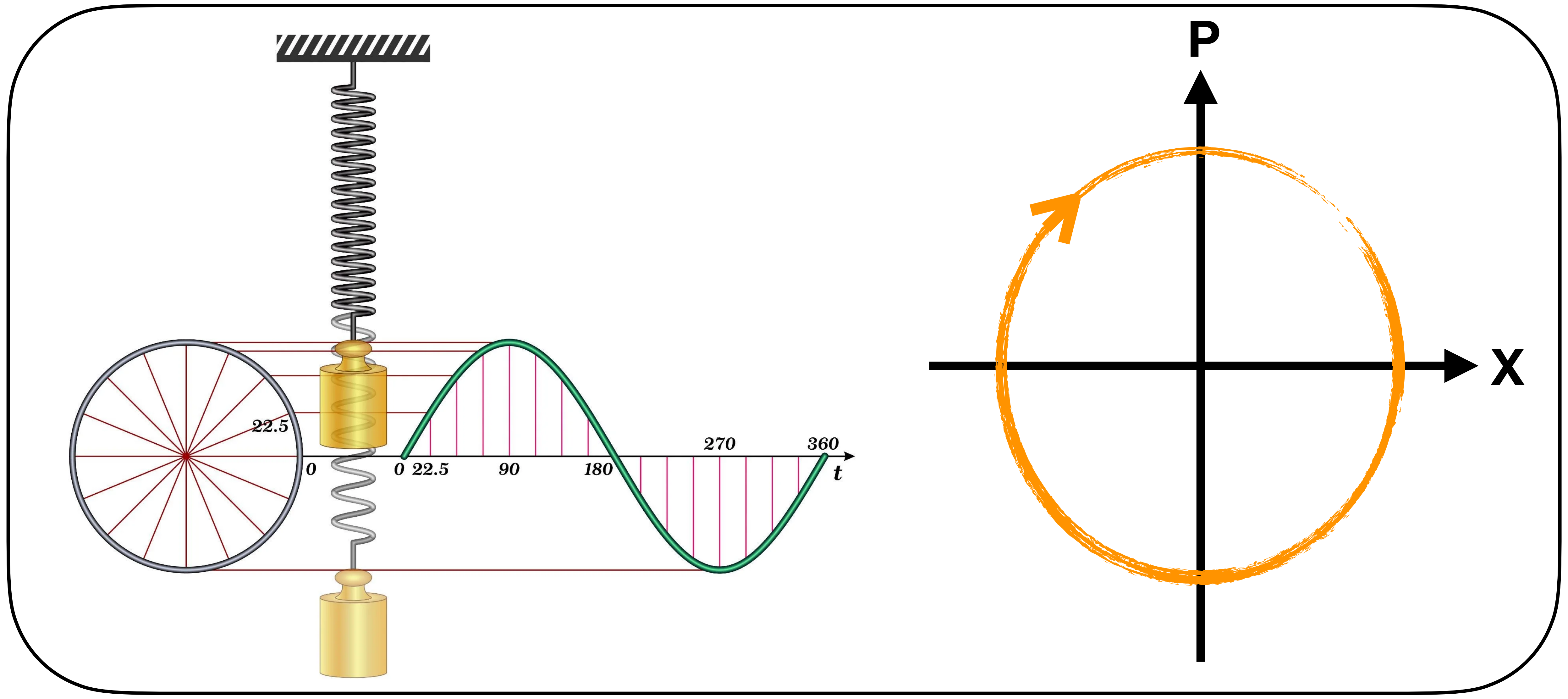
General Qubit Hamiltonian

for time-independent $\vec{\omega}$: $|\Psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\Psi(t=0)\rangle$

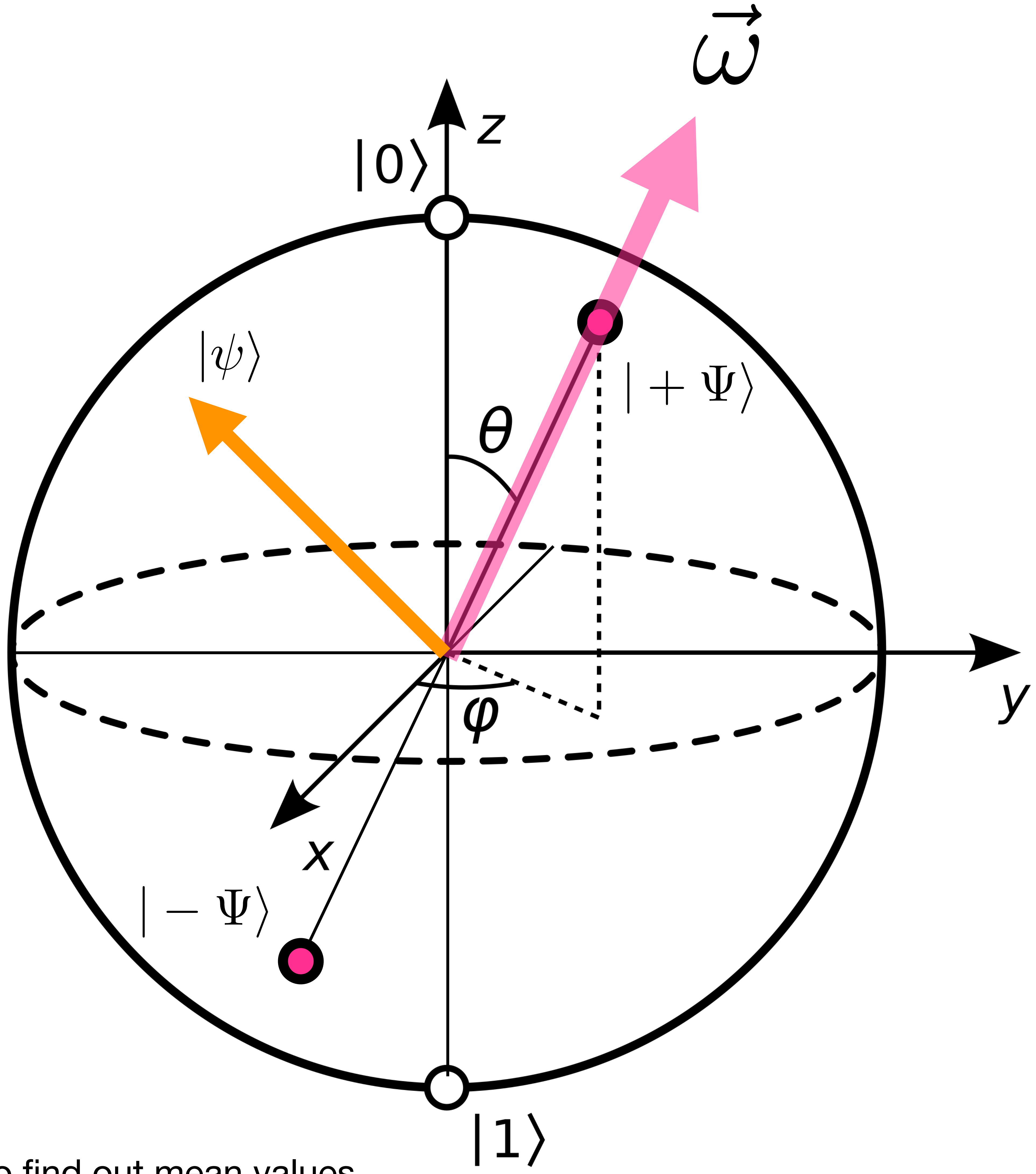
rotation around $\vec{\omega}$

at a constant rate ω

compare to motion of **classical** harmonic oscillator!



Quantum measurement: qubit is a unit-length arrow, on average!



To find out mean values, we must have many qubits prepared in the same initial state…

JUST MATH

Define a direction $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$

Define a hermitian "projection"operator $\hat{\Omega}$

$\hat{\Omega} = \omega_x \hat{X} + \omega_y \hat{Y} + \omega_z \hat{Z}$

$\hat{\Omega} = \omega|+\Psi\rangle\langle+\Psi| - \omega|-\Psi\rangle\langle-\Psi|$

Eigenstates: $|\pm\Psi\rangle$ Eigenvalues: $\pm\omega$

$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$

$\omega_z = \omega\cos\theta$

$\omega_x = \omega\sin\theta\cos\varphi$

$\omega_y = \omega\sin\theta\sin\varphi$

CRAZY WORLD WE LIVE IN

ALLOVED QUERY:

qubit in state $|\psi\rangle$, $?\hat{\Omega}$

Reading $+\omega$ with probability $|\langle+\Psi|\psi\rangle|^2$
Qubit state reset to $|+\Psi\rangle$

Reading $-\omega$ with probability $|\langle-\Psi|\psi\rangle|^2$
Qubit state reset to $|-\Psi\rangle$

mean of $?\hat{\Omega} = \langle\hat{\Omega}\rangle = \langle\psi|\hat{\Omega}|\psi\rangle$

Verify this

$$\left(\langle+\Psi|\hat{X}|+\Psi\rangle, \langle+\Psi|\hat{Y}|+\Psi\rangle, \langle+\Psi|\hat{Z}|+\Psi\rangle\right) = \vec{\omega}/\omega$$

$$\left(\langle-\Psi|\hat{X}|-\Psi\rangle, \langle-\Psi|\hat{Y}|-\Psi\rangle, \langle-\Psi|\hat{Z}|-\Psi\rangle\right) = -\vec{\omega}/\omega$$

Rabi oscillations in a **periodically-driven** qubit

“Laboratory” frame

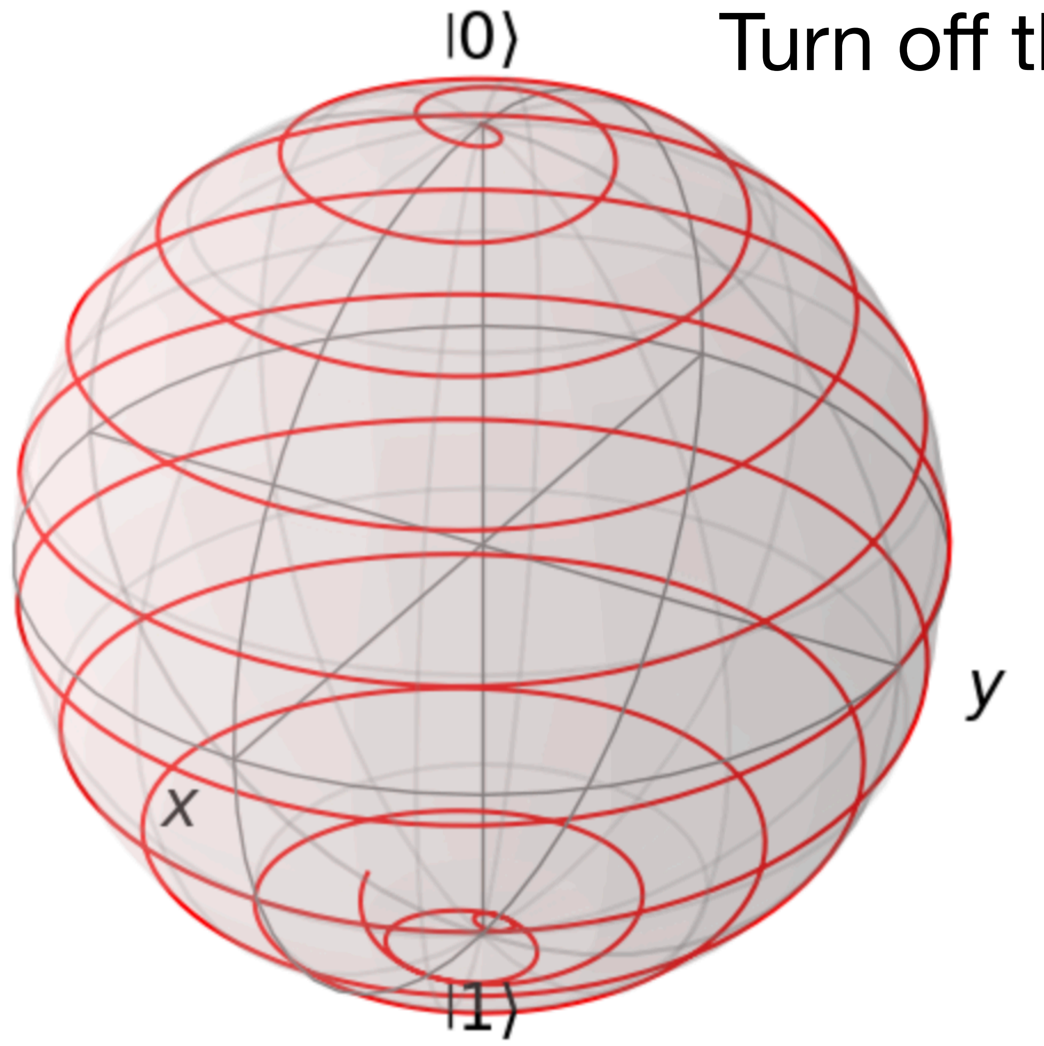
$$\frac{\partial|\Psi\rangle}{\partial t} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle$$

$$\hat{H}/\hbar = -\frac{1}{2}\omega\hat{Z} - (g\cos\omega_d t)\hat{X}$$

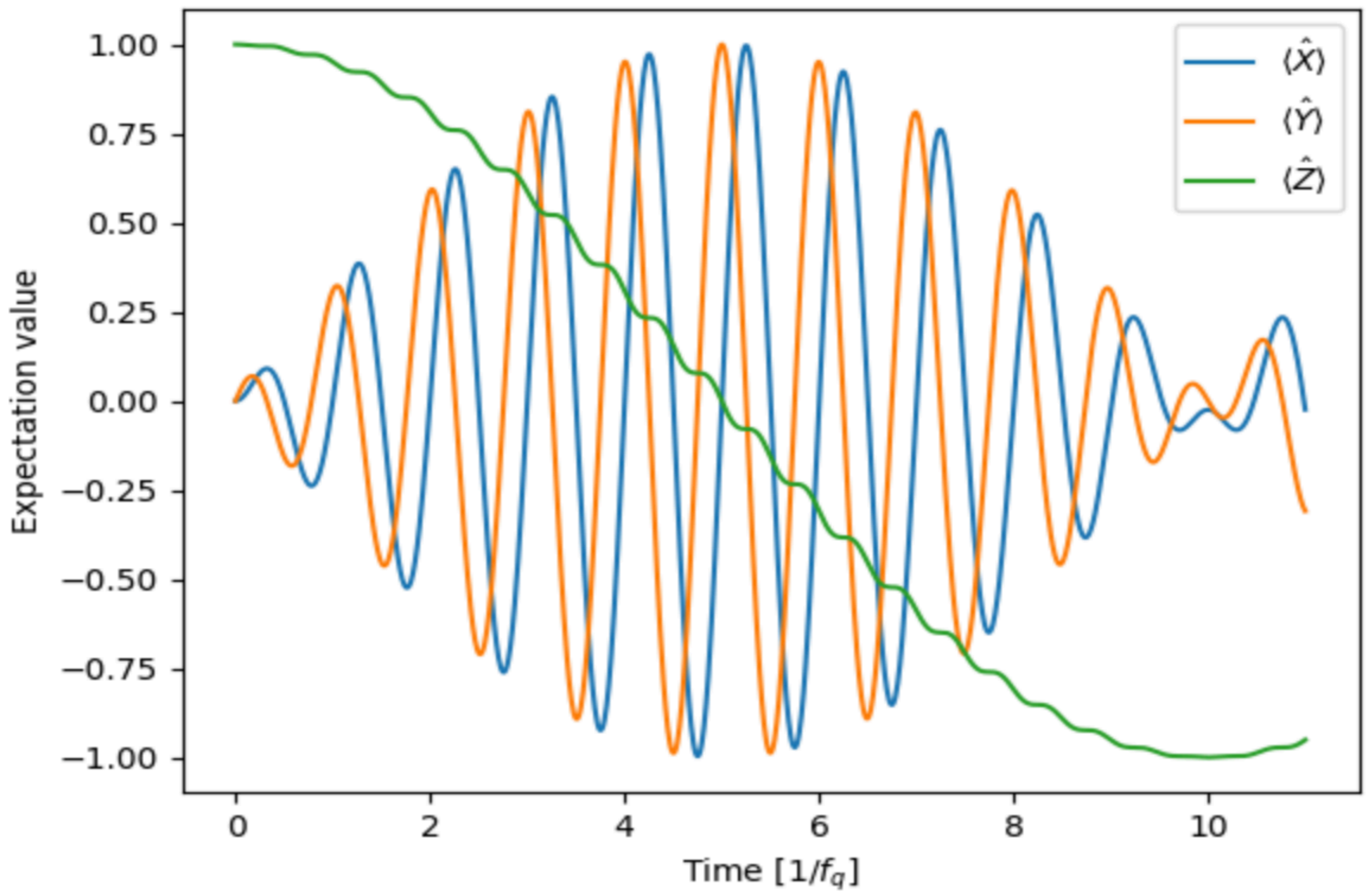
$$g \ll \omega \qquad \omega_d - \omega = \Delta \ll \omega$$

Resonance drive: $\Delta = 0$

Check case $\Delta \approx g$



Turn off the brain, duly solve in QuTip in the lab frame



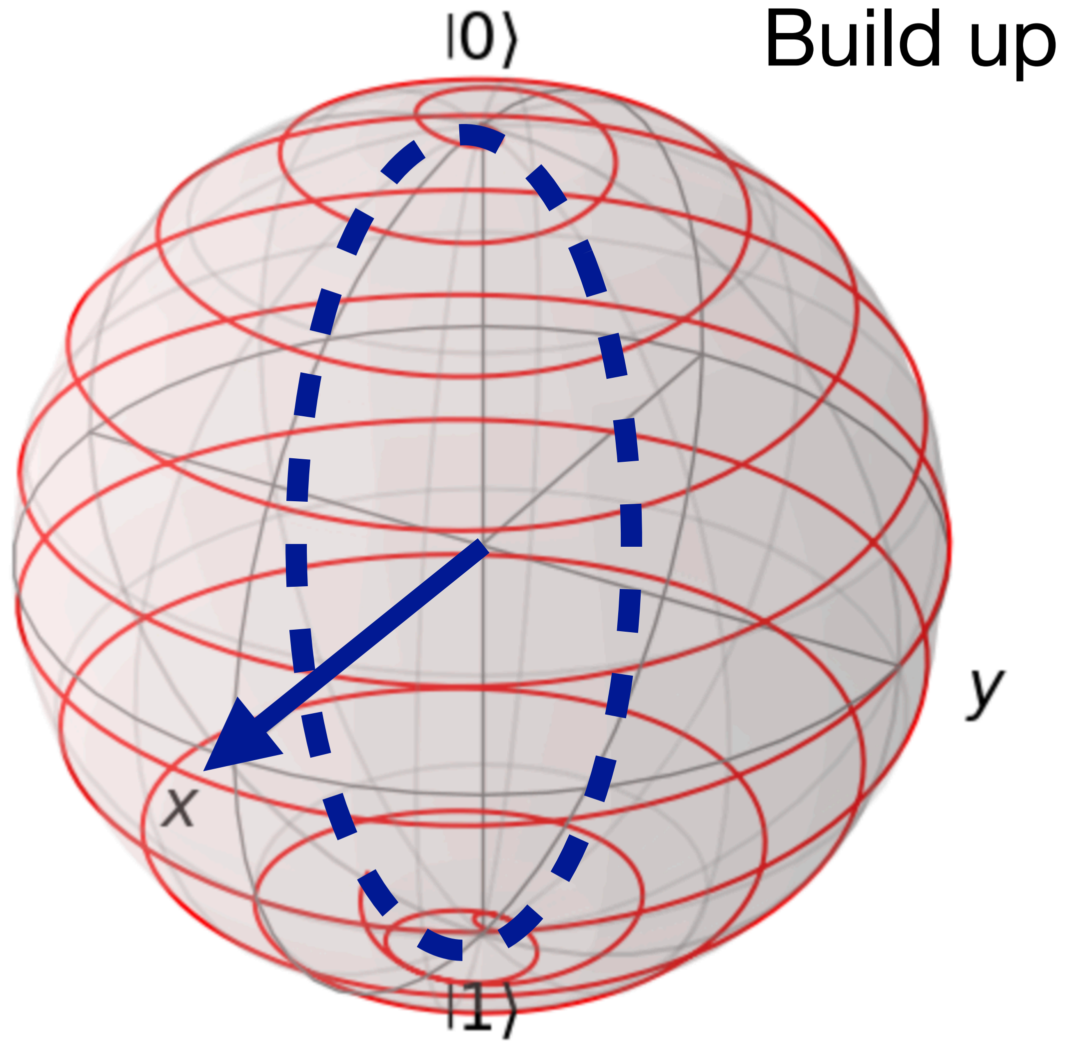
“Rotating” frame (unwinding around **Z**)

$$|\tilde{\Psi}\rangle = \hat{U}(t)|\Psi\rangle$$

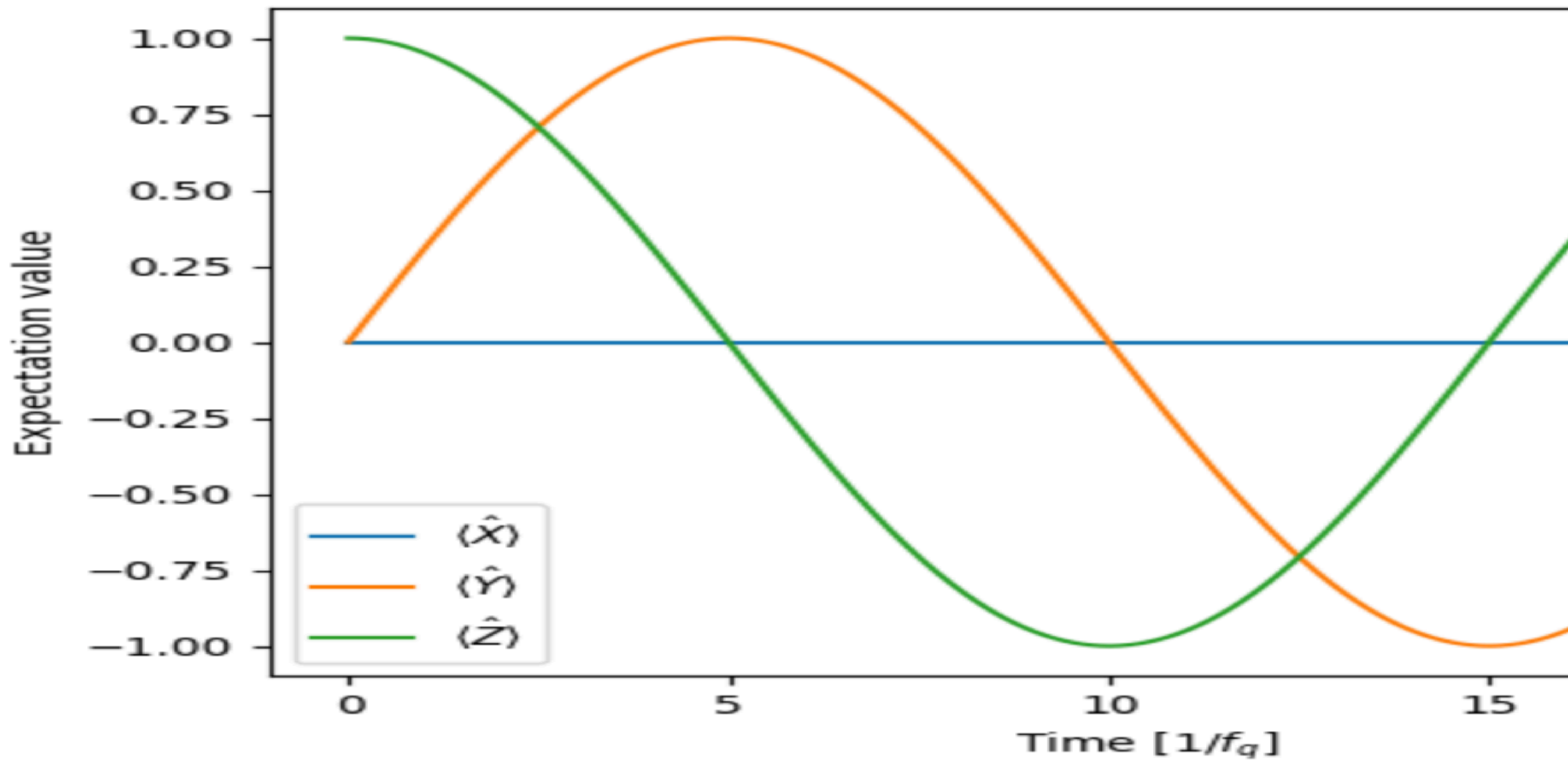
$$\hat{U}(t) = \exp(-i\omega t\hat{Z}/2)$$

$$\frac{\partial|\tilde{\Psi}\rangle}{\partial t} = -\frac{i}{\hbar}\hat{\tilde{H}}|\tilde{\Psi}\rangle \qquad \hat{\tilde{H}}/\hbar \approx -\frac{g}{2}\hat{X}$$

Oscillations between $|0\rangle$ and $|1\rangle$ at a rate g, period $2\pi/g$



Build up intuition in the rotating frame



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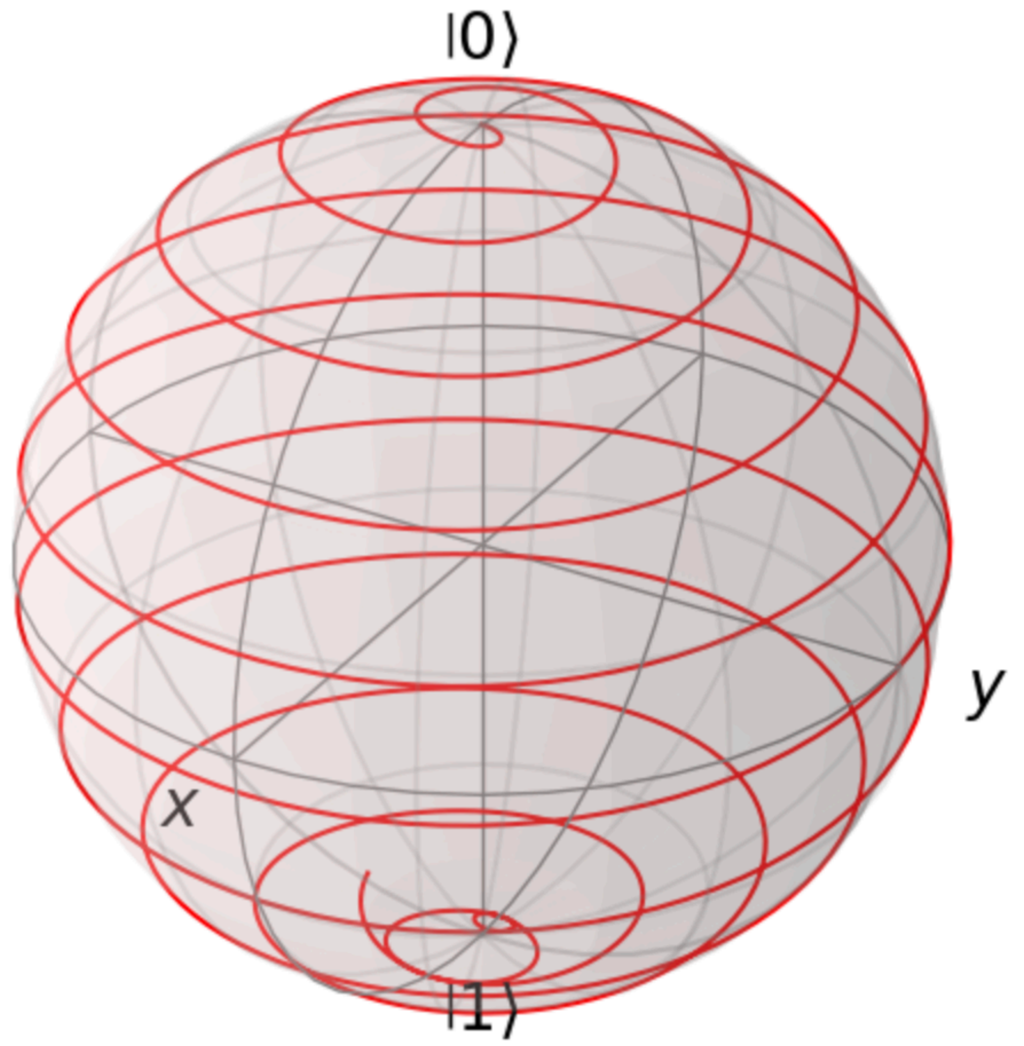
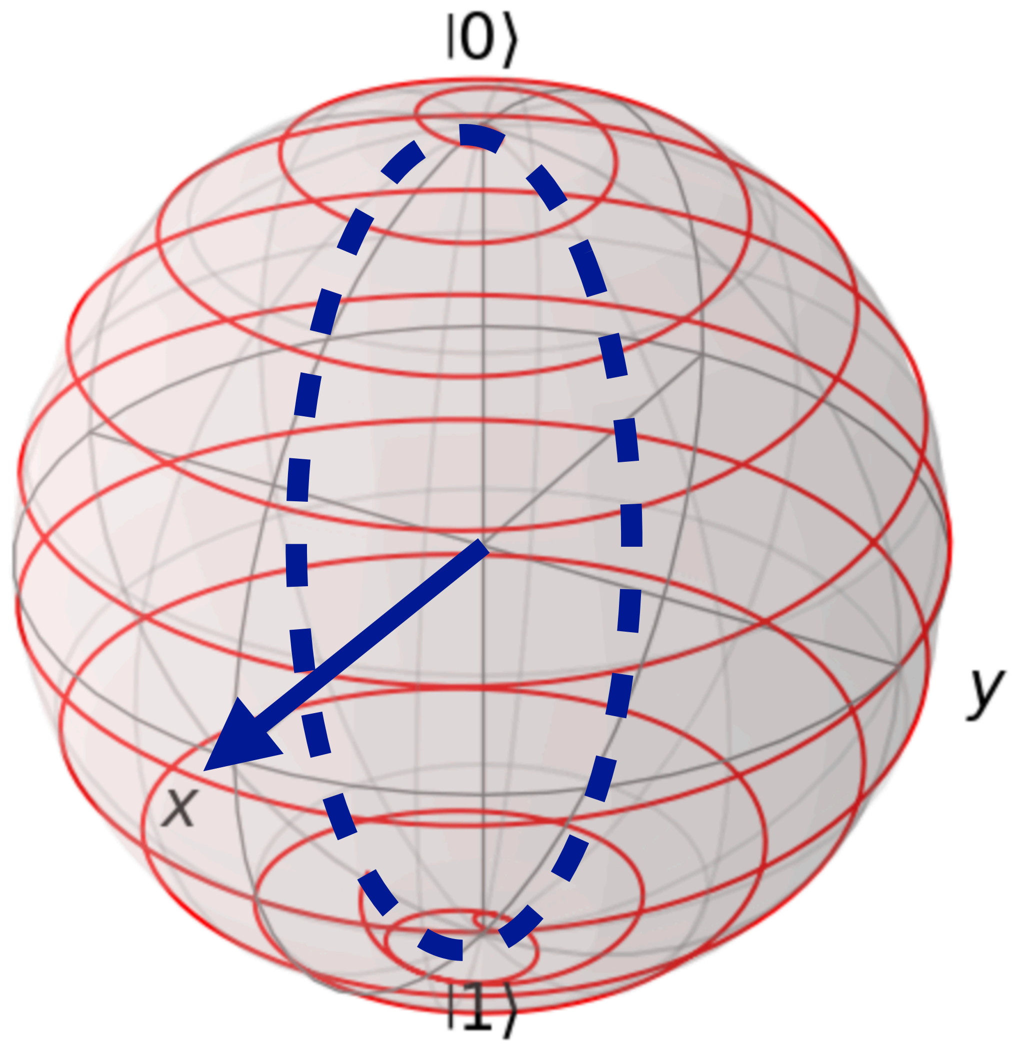
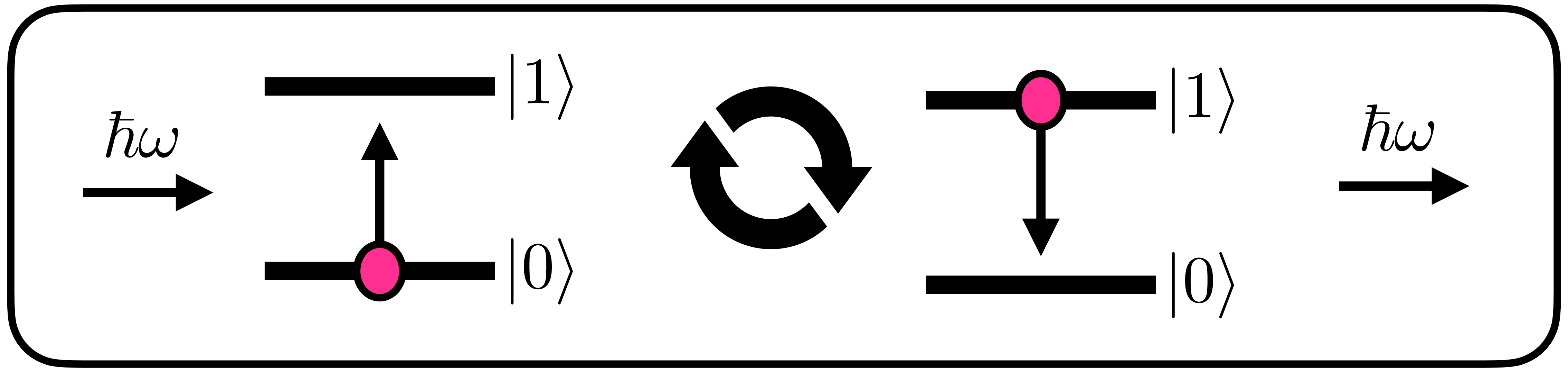




what really happens



photon **in**

photon **out**

contrast to resonantly driven classical harmonic oscillator