3.A PROOF OF THE REYNOLDS TRANSPORT THEOREM

3.B DERIVATION OF THE OSEEN TENSOR

There are many ways to solve the Stokes equation, in combination with the incompressibility condition, including series expansions and Laplace and Fourier transformations. In this section we'll use the latter transformation, which translates our partial differential equations into algebraic equations in Fourier space. These algebraic equations are easily solved, and give us the Oseen tensor (the Green's 'function' relating a point force to the resulting fluid flow) in Fourier space, as well as the pressure in Fourier space; the hardest part is actually the back-transform to real space. As there are a number of different conventions for the Fourier transformations, let us be explicit in stating which one we use (forward and back transforms):

$$\tilde{f}(\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot\boldsymbol{q}}, \tag{3.61}$$

$$f(\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^n} \tilde{f}(\boldsymbol{q}) e^{-i\boldsymbol{x}\cdot\boldsymbol{q}}, \tag{3.62}$$

where n is the number of dimensions. Fourier transforms are so useful because they 'translate' derivatives to multiplications with wave vectors:

$$\int \mathrm{d}\boldsymbol{x} \frac{\partial f(\boldsymbol{x})}{\partial x_i} e^{i\boldsymbol{x}\cdot\boldsymbol{q}} = -\int \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}) \frac{\partial e^{i\boldsymbol{x}\cdot\boldsymbol{q}}}{\partial x_i} = -i q_i \tilde{f}(\boldsymbol{q}). \tag{3.63}$$

We now apply the Fourier transform to our set of differential equations, written out in coordinate form:

$$-\partial_i p + \eta \partial_k \partial_k v_i + F_i \delta(\boldsymbol{x}) = 0, \tag{3.64}$$

$$\partial_i v_i = 0, \tag{3.65}$$

which gives (note that the F_i are just numbers, so they remain the same under the Fourier transform; also, using the convention given in (3.61), the Fourier transform of the Dirac delta function is simply 1):

$$i q_i \tilde{p} - \eta q_k q_k \tilde{v}_i + F_i = 0, \tag{3.66}$$

$$-i q_i \tilde{v}_i = 0. \tag{3.67}$$

Note that $q_k q_k = \boldsymbol{q}\cdot\boldsymbol{q} = q^2$ with q the length of $\boldsymbol{q}$. From (3.66) and (3.67) we can easily solve for $\tilde{p}$, by contracting (i.e., taking the dot product) equation (3.66) with q_i, and applying equation (3.67):

$$0 = i q^2 \tilde{p} + q_i F_i,$$

$$\tilde{p} = i \frac{q_j F_j}{q^2}. \tag{3.68}$$

Note that in (3.68) we replaced the summation (aka 'dummy') index i with j, to avoid confusion in the next step: we substitute (3.68) back into (3.66) and solve for $\tilde{v}_i$:

$$0 = -\frac{q_i q_j}{q^2} F_j - \eta q^2 \tilde{v}_i + F_i$$

$$\tilde{v}_i = \frac{1}{\eta q^2} \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) F_i = \tilde{J}_{ij}(\boldsymbol{q}) F_i, \tag{3.69}$$

where we wrote $F_i = \delta_{ij} F_j$ to be able to write the velocity as a tensor contracted with the vector $\boldsymbol{F}$, and introduced the (Fourier transform of) the Oseen tensor $\tilde{J}_{ij}(\boldsymbol{q})$. Equations (3.69) and (3.68) are the solution in Fourier space. To transform them back to real space, we note that $\tilde{p}$ is given by a dot product: that of $\boldsymbol{q}$ and $\boldsymbol{F}$, while $\boldsymbol{v}$ is the contraction of a symmetric two-tensor with the vector $\boldsymbol{F}$. We expect the same result in real space. For the pressure, the only candidate vector is the position vector, so we expect the pressure to take the form

$$p(\boldsymbol{x}) = a(x) \frac{x_i}{x} F_i = \frac{a(x)}{x} \boldsymbol{x}\cdot\boldsymbol{F}, \tag{3.70}$$

where x is the length of the vector $\boldsymbol{x}$, and the function $a(x)$ can depend on x but not on its individual components. Requiring that $p(\boldsymbol{x})$ has the form (3.70), we find that

$$a(x) \frac{x_i}{x} = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3} \frac{q_i}{q^2} e^{-i\boldsymbol{x}\cdot\boldsymbol{q}}. \tag{3.71}$$

To find $a(x)$, we need to evaluate the integral on the right hand side of (3.71); moreover, we need to find the component of x_i/x. Fortunately, the latter is easy: we simply contract both sides of equation (3.70) with x_i/x. The left hand side becomes $a(x)x_ix_i/x^2 = a(x)$, while on the right hand side we get a term of the form x_iq_i, or the dot product of $\boldsymbol{x}$ and $\boldsymbol{q}$, the same as we have in the exponent that comes with the Fourier transform. To evaluate the integral, we choose coordinates of $\boldsymbol{q}$-space such that the $\boldsymbol{x}$ vector lies along the 3-axis. We then go to spherical coordinates, where the polar angle θ is the same as the angle between $\boldsymbol{x}$ and $\boldsymbol{q}$, so we can write $\boldsymbol{x}\cdot\boldsymbol{q} = x_iq_i = xq\cos\theta$. We then find for $a(x)$:

$$a(x) = i\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\int_0^{\pi}\frac{\sin\theta\,\mathrm{d}\theta}{2\pi}\int_0^{\infty}\frac{q^2\,\mathrm{d}q}{2\pi}q\cos\theta e^{-iqx\cos\theta}. \tag{3.72}$$

The integrand in (3.72) does not depend on the azimuthal angle ϕ, so that integral is trivial. The remaining integral is of a more general form, which we'll encounter again, so we give it its own name:

$$g_n(\boldsymbol{x}, f) = \int_0^{\pi}\frac{\sin\theta\,\mathrm{d}\theta}{2\pi}\int_0^{\infty}\frac{\mathrm{d}q}{2\pi}q^n f(\cos\theta)e^{-iqx\cos\theta}, \tag{3.73}$$

where n is an integer and $f(z)$ is any analytical function; in equation (3.72) we have $n = 1$ and $f(z) = z$ (or $f(\cos\theta) = \cos\theta$). Note that we canceled the q^2 from the transition to spherical coordinates with the $1/q^2$ term from the integrand. To evaluate the integral in (3.73), we make two coordinate transformations: $z = \cos\theta$, and $y = qx$, where we note that x, as the length of the position vector, is always positive:

$$g_n(\boldsymbol{x}, f) = \frac{1}{2\pi|x|^{n+1}}\int_{-1}^{1}\mathrm{d}z\,f(z)\int_0^{\infty}\frac{\mathrm{d}y}{2\pi}y^n e^{-iyz}. \tag{3.74}$$

Since $f(z)$ does not depend on y, we have pulled it outside the y integral. That latter integral now strongly resembles an inverse Fourier transform of a one-dimensional variable, only over half the interval. Moreover, we can obtain it from taking successive z derivatives of the inverse Fourier transform of 1:

$$\int_0^{\infty}\frac{\mathrm{d}y}{2\pi}y^n e^{-iyz} = i^n\frac{\partial^n}{\partial z^n}\int_0^{\infty}\frac{\mathrm{d}y}{2\pi}e^{-iyz} = \frac{i^n}{2}\frac{\partial^n}{\partial z^n}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{2\pi}e^{-iyz} = \frac{i^n}{2}\frac{\partial^n}{\partial z^n}\delta(z), \tag{3.75}$$

where $\delta(z)$ is the one-dimensional Dirac delta function of z. Substituting back into equation (3.74) and repeated partial integration gives us $g_n(\boldsymbol{x}, f)$ for arbitrary functions $f(z)$:

$$\begin{aligned} g_n(\boldsymbol{x}, f) &= \frac{i^n}{4\pi|x|^{n+1}}\int_{-1}^{1}\mathrm{d}z\,f(z)\frac{\partial^n}{\partial z^n}\delta(z) \\ &= \frac{(-i)^n}{4\pi|x|^{n+1}}\int_{-1}^{1}\mathrm{d}z\left(\frac{\partial^n f(z)}{\partial z^n}\right)\delta(z) \\ &= \frac{(-i)^n}{4\pi|x|^{n+1}}\left(\frac{\partial^n f(z)}{\partial z^n}\right)_{z=0}. \end{aligned} \tag{3.76}$$

For the pressure, we now substitute $n = 1$ and $f(z) = z$, to get:

$$a(x) = ig_1(\boldsymbol{x}, z) = i\frac{(-i)}{4\pi x^2}(1) = \frac{1}{4\pi x^2}, \tag{3.77}$$

$$p(\boldsymbol{x}) = a(x)\frac{x_i}{x}F_i = \frac{x_i}{4\pi x^3}F_i = \frac{\boldsymbol{x}\cdot\boldsymbol{F}}{4\pi x^3}, \tag{3.78}$$

as given by equation (3.57). To find the velocity profile, we use a similar tactic. We know that the Oseen tensor $J_{ij}(\boldsymbol{x})$ must be symmetric in i and j (because so is its Fourier transform), which limits us to two possible contributions: δ_{ij} and $\frac{x_ix_j}{x^2}$. In general, J_{ij} will be a linear combination of these two 'basis tensors'. However, they are not orthogonal: $\delta_{ij}\frac{x_ix_j}{x^2} = 1$, not 0, which makes determining their coefficients less efficient (we could do it here, but in general, it is much easier to work with orthonormal bases). Instead, we therefore use $\frac{x_ix_j}{x^2}$ and $\frac{1}{\sqrt{2}}\left(\delta_{ij} - \frac{x_ix_j}{x^2}\right)$, which are orthonormal (the factor $1/\sqrt{2}$ comes of course from the fact that $\delta_{ii} = 3$). We write for the Oseen tensor in real space:

$$J_{ij}(\boldsymbol{x}) = b(x)\frac{x_ix_j}{x^2} + \frac{c(x)}{\sqrt{2}}\left(\delta_{ij} - \frac{x_ix_j}{x^2}\right), \tag{3.79}$$

where as with the pressure, $b(x)$ and $c(x)$ are functions of the magnitude of x only. To find them, we contract the inverse Fourier transform of $\tilde{J}_{ij}(\boldsymbol{q})$ with the two basis tensors. Fortunately, all integrals are of the type $g_n(\boldsymbol{x}, f)$ which we already found; now we need $n = 0$ and two functions for $f(z)$: $f(z) = 1$ and $f(z) = z^2$:

$$\begin{aligned} b(x) &= \frac{x_i x_j}{x^2} J_{ij} \\ &= \frac{x_i x_j}{x^2} \int_0^{\pi} \frac{\sin\theta \, \mathrm{d}\theta}{2\pi} \int_0^{\infty} \frac{q^2 \, \mathrm{d}q}{2\pi} \frac{1}{\eta q^2} \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) e^{-i q_k x_k} \\ &= \frac{1}{\eta} \int_0^{\pi} \frac{\sin\theta \, \mathrm{d}\theta}{2\pi} \int_0^{\infty} \frac{\mathrm{d}q}{2\pi} \left(1 - \cos^2\theta \right) e^{-i q x \cos\theta} \\ &= \frac{1}{\eta} \left[g_0(\boldsymbol{x}, 1) - g_0(\boldsymbol{x}, z^2) \right] \\ &= \frac{1}{\eta} \frac{1}{4\pi x}, \end{aligned} \tag{3.80}$$

and

$$\begin{aligned} c(x) &= \frac{1}{\sqrt{2}} \left(\delta_{ij} - \frac{x_i x_j}{x^2} \right) J_{ij} \\ &= \frac{1}{\eta} \frac{1}{\sqrt{2}} \left[(3-1) g_0(\boldsymbol{x}, 1) - \left(g_0(\boldsymbol{x}, 1) - g_0(\boldsymbol{x}, z^2) \right) \right] \\ &= \frac{1}{\eta} \frac{1}{\sqrt{2}} \frac{1}{4\pi x}. \end{aligned} \tag{3.81}$$

The Oseen tensor is thus given by

$$\begin{aligned} J_{ij}(\boldsymbol{x}) &= \frac{1}{\eta} \frac{1}{4\pi x} \left[\frac{x_i x_j}{x^2} + \frac{1}{2} \left(\delta_{ij} - \frac{x_i x_j}{x^2} \right) \right] \\ &= \frac{1}{\eta} \frac{1}{8\pi x} \left(\delta_{ij} + \frac{x_i x_j}{x^2} \right), \end{aligned} \tag{3.82}$$

and for the velocity profile we get

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{J} \cdot \boldsymbol{F} = \frac{1}{8\pi\eta r} \left(\boldsymbol{F} + \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{x^2} \boldsymbol{x} \right), \tag{3.83}$$

as given before in equation (3.56).