

Exercise sheet #1

Problem 1. Both the atmosphere and the oceans are discrete collections of molecules that we can reimagine as a continuum described by a density field $\rho(\mathbf{r}, t)$. What is the spacing between the molecules of air and water for the density fields of the atmosphere and ocean respectively? According to this spacing, can you justify approximating the ocean and atmosphere as continua?

Use the following assumptions:

- mass density of air: $\rho_{m, \text{air}} \approx 1 \text{ kg m}^{-3}$
molar mass of air $M_{\text{air}} \approx 30 \text{ g mol}^{-1}$
- mass density of water: $\rho_{m, \text{water}} = 1 \text{ g cm}^3$
molar mass of water $M_{\text{water}} \approx 20 \text{ g mol}^{-1}$.
- Avogadro's number $N_A \approx 6 \times 10^{23} \text{ mol}^{-1}$.

Solution: The density can be described either in terms of molecular counts (number of particles per volume) or mass (quantity of mass units per volume). Here, we are interested in the spacing of the molecules that we coarse-grain in the density field $\rho(\mathbf{r}, t)$. The volume taken up per molecule is given by $V/N = 1/\rho$, which implies in turn that the spacing between molecules is given by

$$\text{molecular spacing} = \frac{1}{\rho^{1/3}}$$

We first convert the mass densities into number densities by dividing by the mass per molecule. Using the values provided in the problem statement, we find

$$\rho_{\text{air}} = 1000 \frac{\text{g}}{\text{m}^3} \times \frac{1 \text{ mol}}{30 \text{ g}} \times 6 \times 10^{23} \frac{\text{molecules}}{\text{mol}} \approx 2 \times 10^{25} \frac{\text{molecules}}{\text{m}^3}.$$

We can now use this density to estimate the spacing of molecules in the atmosphere, namely,

$$d_{\text{air}} = \frac{1}{\sqrt[3]{2 \times 10^{25}}} m = \frac{1}{\sqrt[3]{20 \times 10^{24}}} m = \frac{1}{\sqrt[3]{20}} 10^{-8} m \approx 4 \text{ nm}.$$

We can use the atmospheric estimate to work out the mean spacing of molecules in water by noting that

$$d_{\text{water}} = \frac{1}{\rho_{\text{water}}^{1/3}} = \frac{1}{(1000 \rho_{\text{air}})^{1/3}} = \frac{1}{10} d_{\text{air}}.$$

Molecules in the atmosphere are spaced a few nanometers apart, while in water the spacing is on the order of a few tenths of a nanometer. These separations are tiny compared to the macroscopic length scales relevant for describing atmospheric and oceanic flows. For this reason, treating the ocean and atmosphere as continuum media is well justified. Continuum theory, however, fails when the characteristic length scales of interest become comparable to the microscopic spacing between molecules. At the same time, one cannot simply average over arbitrarily large volumes: the averaging length scale must remain much smaller than the scale over which the macroscopic fields vary, so that local variations are still resolved. \square

Problem 2. Find the bulk modulus of an ideal gas that is compressed by a hydraulic strain. Recall that for an ideal gas: $PV = Nk_B T$

Solution: Using the definition of the bulk modulus for a hydraulic strain:

$$\Delta P = -K \frac{\Delta V}{V}$$

we have

$$K = -V \frac{\Delta P}{\Delta V} = -V \frac{dP}{dV}$$

For an ideal gas we have

$$PV = Nk_B T \quad \Rightarrow \quad P = \frac{N}{V} k_B T,$$

$$\frac{dP}{dV} = -\frac{N}{V^2} k_B T.$$

So the bulk modulus for an ideal gas is given by

$$K = -V \frac{dP}{dV} = nk_B T,$$

where $n = N/V$.

□

Problem 3. In this problem, we'll analyze the Poisson effect for the extension of a homogeneous and isotropic elastic body which originally has the shape of a cube with sides of length L . We choose coordinates such that the origin is at the center of the cube, and the axes are aligned with its sides. We apply a tensile stress along the x axis, causing the cube to deform into a rectangular block, with the dimension along the x axis now $L + 2\Delta L$, and along the y and z axes now $L - 2\Delta L'$.

- a) Consider a point inside the cube at position $\mathbf{r} = (x, y, z)$. After the deformation, the point has moved to $\mathbf{r}' = \mathbf{r} + d\mathbf{r}$. As the origin stays fixed and the sides of the cube move, the deformation is not constant throughout the block. Argue why the local strain is given by

$$d\gamma_x = \frac{dx}{x}, \quad d\gamma_y = -\frac{dy}{y}, \quad d\gamma_z = -\frac{dz}{z}.$$

where we've taken the deformations dx , dy and dz to be all positive.

- b) By definition of the Poisson ratio ν (and symmetry in y and z), we have $-\nu d\gamma_x = d\gamma_y = d\gamma_z$. As our material is homogeneous, we can integrate the resulting differential equation, to get

$$-v \int_L^{L+\Delta L} \frac{dx}{x} = - \int_{L-\Delta L'}^L \frac{dy}{y} = \int_L^{L-\Delta L'} \frac{dy}{y}$$

(We're leaving out the integral over z as it is identical to the one over y). Carry out the integration, to find an equation containing ν and the ratios $\Delta L/L$ and $\Delta L'/L$.

- c) Using the expansion $(1+x)^n = 1 + nx + O(x^2)$ (valid for small x , and even for n not an integer and/or negative), simplify the relation you found in (b) to get the approximate expression for the Poisson ratio we use mostly in practice.

Solution: a) The deformation is zero at the origin, and respectively ΔL and $-\Delta L'$ at the extending and contracting surfaces. As the body is homogeneous and isotropic, the deformation will be linear in the distance to the origin, and therefore the strain in each of the three cardinal directions is the ratio of the deformation to the distance to the origin (with appropriate signs), as given by $d\gamma_x = \frac{dx}{x}$, $d\gamma_y = -\frac{dy}{y}$, $d\gamma_z = -\frac{dz}{z}$.

- b) The integrals give us logarithms. Leaving out the z part (identical to the y one), we have

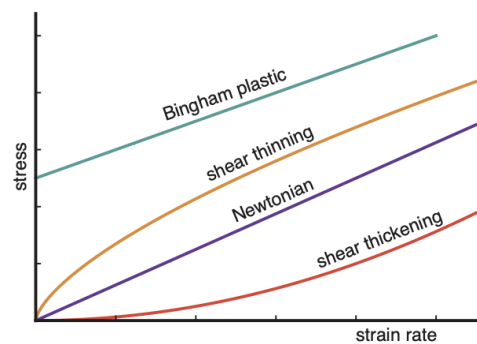
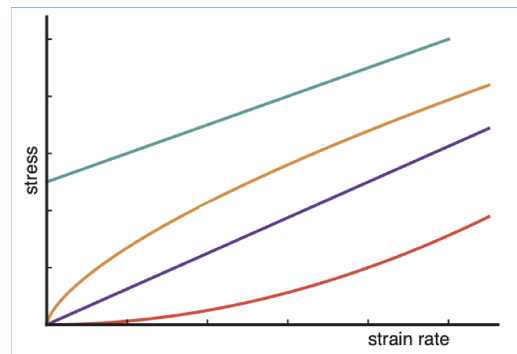
$$\begin{aligned}
 -v \log \left(\frac{L + \Delta L}{L} \right) &= \log \left(\frac{L - \Delta L'}{L} \right) \\
 \log \left[\left(1 + \frac{\Delta L}{L} \right)^{-v} \right] &= \log \left(1 - \frac{\Delta L'}{L} \right) \\
 \left(1 + \frac{\Delta L}{L} \right)^{-v} &= 1 - \frac{\Delta L'}{L}
 \end{aligned}$$

(c) We have

$$\begin{aligned}
 \left(1 - \frac{\Delta L'}{L} \right)^{-v} &\approx 1 - v \frac{\Delta L'}{L} = 1 - \frac{\Delta L'}{L} \\
 v &\approx \frac{\Delta L'}{\Delta L}
 \end{aligned}$$

□

Problem 4. The following plot shows the stress-strain relation for different fluids. Which of the curves describes: a newtonian fluid, a shear thinning fluid, a shear thickening fluid and a Bingham plastic (a type of non-Newtonian fluid that behaves like a solid until a certain minimum stress is applied, and then flows like a viscous fluid).



Solution:

□