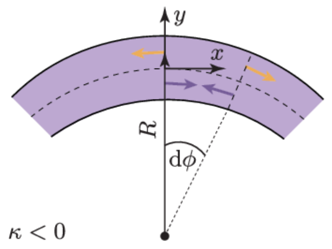
Exercise sheet #7

Problem 1. Derivation of the bending energy of a bent rod/sheet



a) Argue for the situation sketched in the figure that for a sheet bent in the x direction only, the fact that there is no external pressure applied to the surface of the sheet implies that at the upper and lower side $\sigma_{yi} = 0$ for $i = x, y, z$.

b) Argue that since in particular $\sigma_{yy} = 0$ at the upper and lower sides, to lowest order in the curvature we must have $\sigma_{vy} = 0$ everywhere within the sheet.

c) Use the linear stress-strain equation $\sigma_{ij} = K\gamma_{kk}\delta_{ij} + 2G(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta ij)$ for $\sigma_{yy} = 0$ to derive $\gamma_{yy} \approx \frac{-\nu}{1-\nu}\frac{y}{R}$.
Hint: It is easiest to express K and G immediately in terms of E_γ and ν using equations $K = \frac{E_\gamma}{3(1-2\nu)}$ and $G = \frac{E_\gamma}{2(1+\nu)}$.

d) Show with these results that $\sigma_{xx} = \frac{E_\gamma y}{(1-\nu^2)R}$ and that $\sigma_{ij}u_{ij} = E_\gamma y^2 / \left(1 - \nu^2\right) R^2$.

Solution: a) The fact that there is no pressure applied to upper and lower surfaces imply that the perpendicular force due to strain on the surface must be zero, which is

$$\sigma_{ik}n_k = 0$$

where n_k is the normal vector to the surface. Since the sheet is bent in x direction in this case, and the bending is small, $\hat{n} \approx \hat{y}$, which yields

$$\sigma_{iy} = \sigma_{yi} = 0 \quad \text{for all } i.$$

b) This simply follows from the fact that the sheet is very thin. If $\sigma_{yy} = 0$ on the whole boundary, then within a small thickness, we do not expect a wild variation. Hence it is reasonable to assume $\sigma_{yy} \approx 0$ anywhere in the plate. We can make this more precise as follows: in lowest order of approximation, components of the stress vary linearly in y. This will also be true for σ_{yy}, and the only linear function which is zero at both boundaries is zero everywhere within the sheet.

c) Following the hint, we have

$$\sigma_{yy} = K\left(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}\right) + 2G\left(\gamma_{yy} - \frac{1}{3}\left(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}\right)\right) = 0$$

Inserting $K = E_\gamma/3(1-2\nu)$ and $G = E_\gamma/2(1+\nu)$, we get

$$\sigma_{yy} = \frac{E_\gamma}{3(1-2\nu)}\left(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}\right) + \frac{E_\gamma}{(1+\nu)}\gamma_{yy} - \frac{E_\gamma}{3(1+\nu)}\left(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}\right) = 0.$$

Due to how bending is applied and hence the symmetry of the system, $\gamma_{zz} = 0$; this yields

$$\left[\frac{1}{2-\nu}+\frac{2}{1+\nu}\right]\gamma_{yy}=\left[\frac{1}{1+\nu}-\frac{1}{1-2\nu}\right]\gamma_{xx}$$

which gives,

$$3(1-\nu)\gamma_{yy}=-3\nu\gamma_{xx}\quad\Rightarrow\quad\gamma_{yy}=-\frac{\nu}{1-\nu}\gamma_{xx}=-\frac{\nu}{1-\nu}\frac{y}{R}$$

d) For σ_{xx} we have,

$$\begin{aligned}\sigma_{xx}&=K\left(\gamma_{xx}+\gamma_{yy}\right)+2\mu\left(\gamma_{xx}-1/3\left(\gamma_{xx}+\gamma_{yy}\right)\right)=\frac{E_\gamma}{3}\left[\frac{1}{1-2\nu}+\frac{2}{1+\nu}\right]\gamma_{xx}+\frac{E_\gamma}{3}\left[\frac{1}{1-2\nu}-\frac{1}{1+\nu}\right]\gamma_{yy}\\&=\frac{E_\gamma}{3(1+\nu)(1-2\nu)}\left[3(1-\nu)\gamma_{xx}+3\nu\gamma_{yy}\right]=\frac{E_\gamma}{(1+\nu)(1-2\nu)}\left[(1-\nu)\gamma_{xx}-\frac{\nu^2}{1-\nu}\gamma_{xx}\right]\\&=\frac{E_\gamma}{(1+\nu)(1-2\nu)}\left[\frac{1-2\nu+\nu^2-\nu^2}{1-\nu}\right]\gamma_{xx}=\frac{E_\gamma}{(1-\nu^2)}\gamma_{xx}=\frac{E_\gamma y}{(1-\nu^2)\,R}\end{aligned}$$

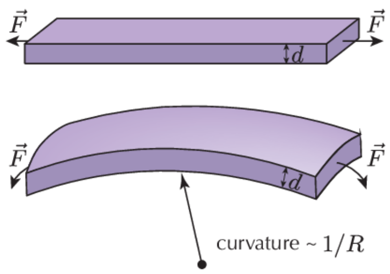
From our results in part a and applying the same logic of part b to all σ_{yi}, we find

$$\sigma_{ij}\gamma_{ij}=\sigma_{xy}\gamma_{xy}+\sigma_{yx}\gamma_{yx}+\sigma_{xx}\gamma_{xx}+\sigma_{yy}\gamma_{yy}\approx\sigma_{xx}\gamma_{xx}=\frac{E_\gamma y^2}{(1-\nu^2)\,R^2}$$

□

Problem 2. Use dimensional analysis to argue why it's much easier to bend than to stretch a sheet of paper. Hint: Consider a sheet or rod of thickness d. Compare the scaling of stretching and bending energies.

Solution: To approach this with a dimensional argument we compare the stretching and bending energy of a sheet of thickness d, as sketched in the figure below.



How will the stretching energy per unit area scale with d? First note that when we deform a straight sheet or rod by stretching it, the energy will increase quadratically in the strain. Likewise, if we bend it the energy must increase quadratically in the bending curvature. Indeed, if there were a linear term, the original object would have been able to lower its energy by spontaneously stretching, contracting or bending. So the deformation energy must be quadratic in the strain and in the curvature. Since strain is dimensionless, and since the dimension of Young's modulus is force per unit area or energy per unit volume, dimensionally the stretching energy must be

$$\frac{E_{\text{stretch}}}{\text{area}}\simeq E_\gamma(\text{ strain })^2 d,$$

that is, it must scale linearly with the thickness d. This is what we know from daily experience-two sheets are twice as strong as one. But intuitively we also know that the force needed to bend a sheet or a rod depends much more on the thickness. Indeed, as the bending energy will be proportional to the square of the curvature (with dimension 1/length) we must have on dimensional grounds

$$\frac{E_{\text{bend}}}{\text{area}} \simeq E_\gamma(\text{ curvature })^2 d^3 \simeq E_\gamma \frac{d^3}{R^2}$$

with R the radius of curvature. You are undoubtedly roughly familiar with this effect from daily experience: if you screw or glue two wooden beams together, they are not twice as resistant to bending deformations, but eight times as resistant.

The important lesson from this comparison is that sufficiently thick sheets and rods for that matter are very rigid and they resist bending, while bending is the low-energy deformation of thin ones.

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Problem 3. In biological systems, elastic materials are often constructed as networks of elastic filaments. A good example is the spectrin network underlying the plasma membrane of a red blood cell, which has a roughly triangular structure. In this problem, we'll study an (ideal) two-dimensional triangular network of springs. Each spring has spring constant k, rest length s_0, and potential energy $V(s) = \frac{1}{2}k(s-s_0)^2$. Note that because we are dealing with a two-dimensional system here, we will use the free energy not per unit volume but per unit area; the two-dimensional version of Hooke's law reads

$$\sigma_{ij} = \frac{1}{2}K_2\gamma_{kk}\delta_{ij} + G_2\left(\gamma_{ij} - \frac{1}{2}\gamma_{kk}\delta_{ij}\right),$$

where K_2 and G_2 are the two-dimensional bulk and shear modulus.

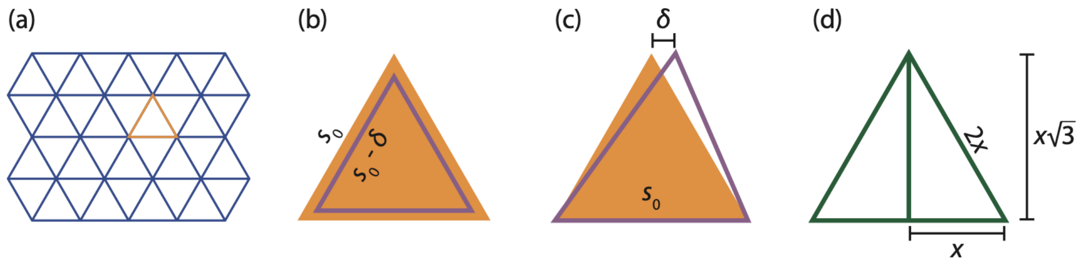

Figure 2.1: Triangular network of springs, as a model for the e.g. the spectrin network in a red blood cell. (a) Configuration of the equilateral triangles. (b) Pure compression by decreasing the length of each spring from s_0 to $s_0 - \delta$. (c) Pure shear by moving the top vertex of a triangle by a distance δ in the direction parallel to the bottom line. (d) Some geometric properties of an equilateral triangle.

a) Suppose we apply a pure compression to our triangular spring network, starting from its equilibrium configuration. The compression results in shortening the length of each spring by a small amount $\delta = s_0 - s$ (see figure 2.1b). Express the change in potential energy per triangle, ΔV, in terms of δ. Note that each spring is shared by two triangles.

b) By dividing your expression in (a) by the area per triangle (in the equilibrium configuration), find the change in the free energy density Δf for the small compression.

c) Express the strain tensor for the pure compression in terms of s_0 and δ. You may ignore any higherorder terms in δ.

d) Substitute the expression for the strain tensor in the expression of the free energy for a continuous material, $\Delta f = \frac{1}{4}\sigma_{ij}\gamma_{ij}$, to obtain an expression for Δf in s_0 and (to lowest order in) δ. You can use the 2D version of Hooke's law from equation $V(u) = -\int F(u)\mathrm{d}u = \frac{1}{2}ku^2$ here. You can use the tensor version of Hooke's law to relate the stress and the strain, but note that here we have a 2D material whereas the expressions in equation $\sigma_{ij} = \frac{1}{2}K\gamma_{kk}\delta_{ij} + G\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right)$ is for 3D.

e) Compare the expression at (d) with your answer at (b) to show that the bulk modulus of the spring network is given by $K_2 = \frac{1}{2}\sqrt{3}k$.

f) From the pure shear deformation in figure 2.1c, show that the shear modulus of the network is given by $G_2 = \frac{1}{4}\sqrt{3}k$.

Solution: a) We have $V(s_0 - \delta) = \frac{1}{2}k\delta^2$ per spring, three springs per triangle, and each spring shared by two triangles, so an average of 3/2 spring per triangle . The change in energy per triangle is thus given by $\Delta V = \frac{3}{2}\frac{1}{2}k\delta^2 = \frac{3}{4}k\delta^2$.

b) The triangles have equilibrium area $\frac{1}{2}s_0\left(\frac{1}{2}\sqrt{3}s_0\right) = \frac{1}{4}\sqrt{3}s_0^2$. The change in energy density per triangle is thus given by

$$\Delta f = \frac{\Delta V}{A} = \frac{\frac{3}{4}k\delta^2}{\frac{1}{4}\sqrt{3}s_0^2} = \sqrt{3}k\frac{\delta^2}{s_0^2}$$

c) The trace of the strain tensor gives the (relative) change in volume, so we have

$$\mathrm{Tr}\left(\gamma_{ij}\right) = 2\frac{\Delta A}{A} = \frac{\left(s_0 - \delta\right)^2 - s_0^2}{s_0^2} \approx -4\frac{\delta}{s_0}$$

Now by symmetry (the network shrinks as much in the horizontal as in the vertical direction), and by virtue of the fact that our strain is a pure compression, we have

$$\gamma_{ij} = \frac{1}{2}\,\mathrm{Tr}\left(\gamma_{ij}\right)\delta_{ij} = -2\frac{\delta}{s_0}\delta_{ij}$$

d) The stress tensor is given by Hooke's law (2D version):

$$\sigma_{ij} = \frac{1}{2}K_2\gamma_{kk}\delta_{ij} = -2\frac{\delta}{s_0}K_2\delta_{ij}$$

where we left out the shear term as it equals zero . Substituting the stress and strain in the expression for the free energy, we get

$$\Delta f = \frac{1}{4}\gamma_{ij}\sigma_{ij} = \frac{1}{2}\left(-\frac{\delta}{s_0}\right)\delta_{ij}\left(-2\frac{\delta}{s_0}\right)K\delta_{ij} = \frac{\delta^2}{s_0^2}K_2\delta_{ij}^2 = 2\frac{\delta^2}{s_0^2}K_2$$

e) Comparing the equations obtained in d and b, we can immediately read off that $\sqrt{3}k = 2K_2$, or $K_2 = \frac{1}{2}\sqrt{3}k$.

f) The steps we need to take for the shear deformation are the same as the ones we took in (a)-(e) for the compression. Note that under the pure shear, the area of the triangle does not change. Moreover, the length of the bottom spring (or any horizontal spring) is unchanged. For the lengths of the left and right spring in figure 2.1c, we get

$$l_{\text{left}} = \sqrt{\left(\frac{1}{2}s_0 + \delta\right)^2 + \left(\frac{1}{2}\sqrt{3}s_0\right)^2} = \sqrt{s_0^2 + s_0\delta + \delta^2} \approx s_0 + \frac{1}{2}\delta,$$
$$l_{\text{right}} = \sqrt{\left(\frac{1}{2}s_0 - \delta\right)^2 + \left(\frac{1}{2}\sqrt{3}s_0\right)^2} = \sqrt{s_0^2 - s_0\delta + \delta^2} \approx s_0 - \frac{1}{2}\delta.$$

Substituting the spring lengths in the potential energy for a triangle, we get

$$\Delta V_{\text{shear}} = \frac{1}{2}\left[\frac{1}{2}k\left(\frac{1}{2}\delta\right)^2 + \frac{1}{2}k\left(-\frac{1}{2}\delta\right)^2\right] = \frac{1}{8}k\delta^2.$$

For the change in free energy per unit area, we then get

$$\Delta f = \frac{\Delta V}{A} = \frac{\frac{1}{8}k\delta^2}{\frac{1}{4}\sqrt{3}s_0^2} = \frac{1}{6}\sqrt{3}k\left(\frac{\delta}{s_0}\right)^2$$

The deformation is a pure shear, of magnitude

$$\frac{\Delta x}{\Delta y} = \frac{\delta}{\frac{1}{2}\sqrt{3}s_0}$$

and for the strain tensor we thus have

$$\gamma_{ij} = \frac{2}{3}\sqrt{3}\frac{\delta}{s_0}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The stress tensor follows from the two-dimensional version of Hooke's law, equation (2.26):

$$\sigma_{ij} = \frac{2}{3}\sqrt{3}G_2\frac{\delta}{s_0}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For the change in free energy per unit area we then get

$$\Delta f = \frac{1}{4}\gamma_{ij}\sigma_{ij} = \frac{2}{3}G_2\left(\frac{\delta}{s_0}\right)^2$$

Comparing both equations for the free energy, we can read off that

$$G_2 = \frac{1}{4}\sqrt{3}k = \frac{1}{2}K_2$$

□